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Revision of $^{212}_{83}$Bi γ-Spectrum by Means of the γ-α Coincidences.

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(ricevuto il 9 Luglio 1957)

Summary. — The fast-slow coincidences between the $^{212}_{83}$Bi γ-rays and the $^{212}_{84}$Po α-particles allow to separate these γ-rays from those emitted by the other nuclides in equilibrium in the $^{228}_{90}$Th source. We can apply this method thanks to the shortness of the half-life of $^{212}_{84}$Po. From our results we can assign to the $^{212}_{83}$Bi some γ-rays uncertain in previous experimental researches. A decay scheme is proposed.

1. – Introduction.

The results on the $^{212}_{83}$Bi γ-spectrum obtained in magnetic spectrometers by several authors (1) don't agree very well among themselves. In a previous paper (2) we have reported the γ-spectrum of this nuclide obtained by means of a scintillation spectrometer.

Our results (2), though reliable for practically all the spectrum, are un-

(1) C. D. ELLIS: *Proc. Roy. Soc.*, A **143**, 350 (1934); A. I. ALIHANOV and V. I. DŽELEPOV: *Compt. Rend. Acad. Sci. URSS*, **20**, 113 (1938); S. C. CURRAN, P. I. DEE and J. E. STROTHERS: *Proc. Roy. Soc.*, A **164**, 546 (1940); G. D. LATYŠEV and L. A. KULCHITSKY: *Journ. Phys. USSR*, **4**, 515 (1941); J. ITOH and Y. WATASE: *Proc. Phys. Math. Soc. Japan*, **23**, 142 (1941); C. E. MANDEVILLE: *Phys. Rev.*, **62**, 309 (1942); G. D. LATYŠEV: *Rev. Mod. Phys.*, **19**, 132 (1947); A. JOHANSSON: *Ark. Mat. Astr. Fys.*, A **34**, 9, 1 (1947); D. G. E. MARTIN and H. O. W. RICHARDSON: *Proc. Phys. Soc.*, A **63**, 223 (1950); D. E. MULLER, H. C. HOYT, D. J. KLEIN and J. W. DUMOND: *Phys. Rev.*, **88**, 775 (1952); D. G. E. MARTIN and G. PARRY: *Proc. Phys. Soc.*, A **28**, 1177 (1955).

(2) B. CHINAGLIA and F. DEMICHELIS: *Nuovo Cimento*, **4**, 1160 (1956).

certain for some γ-rays. Indeed, since the source consists of $^{228}_{90}$Th in equilibrium with its decay products, some γ-rays of $^{212}_{83}$Bi can be hidden by the prominent peaks of more intense γ-rays emitted by the other nuclides.

$^{212}_{83}$Bi goes, by emitting β-particles, to the various excited states of $^{212}_{84}$Po. $^{212}_{84}$Po decays by α emission in $^{208}_{82}$Pb with a half-life of $3.04 \cdot 10^{-7}$ s.

We could analyze the spectrum of the γ-rays coincident with the α-particles of $^{212}_{84}$Po thanks to the shortness of the half-life of this nuclide. In this manner we point out all and only the γ-rays of $^{212}_{83}$Bi provided the excited states of this nuclide live less than the resolving time of our apparatus.

We report our experimental results on the γ-spectrum and we propose a decay scheme of $^{212}_{83}$Bi.

2. – Experimental apparatus and results.

The experimental set up is the same used in a previous research (³).

The source consists of a very thin layer of $^{228}_{90}$Th, (in equilibrium with its decay products; activity about 1 μc) in order to diminish the influence of absorption of the α-particles in the sample.

The γ detector consists of a NaI(Tl) crystal (2 in. $\times$ 2 in.) followed by a photomultiplier tube (DuMont 6292). Some measurements were performed by using a 1 in. $\times$ 1 in. crystal.

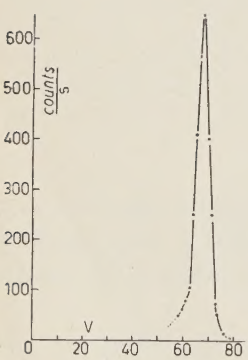

Fig. 1.

The α detector consists of a CsI(Tl) crystal (0.2 mm thick; 10 mm diameter) followed by another photomultiplier tube.

The energy calibration was obtained with the γ-rays of $^{137}_{55}$Cs, $^{60}_{27}$Co and the 2.615 MeV of $^{208}_{81}$Tl and with the α-particles of $^{210}_{84}$Po, $^{214}_{84}$Po and the 8.776 MeV of $^{212}_{84}$Po.

Fig. 1 shows the α spectrum of $^{210}_{84}$Po (energy: 5.298 MeV); the resolution (full width of peak at $\frac{1}{2}$ counting rate) is 5%. The α-particles of $^{212}_{84}$Po were very well separated from the α-particles emitted by the other nuclides.

One of the single channel pulse height analyzer had the window set on the peak corresponding to the α-particles of $^{212}_{84}$Po. The γ-spetcrum in coincidence with the α-particles of this nucilde was swept, with the various positions of the window of the other analyzer, through the range of the γ-spectrum.

(³) F. Demichelis, B. Chinaglia and G. Trivero: *Nuovo Cimento*, in press.

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Fig. 2 shows the single spectrum of $^{228}_{90}$Th in equilibrium with its decay products (continuous line) and the spectrum of the γ-rays in coincidence with the α-particles of $^{212}_{84}$Po (dotted line).

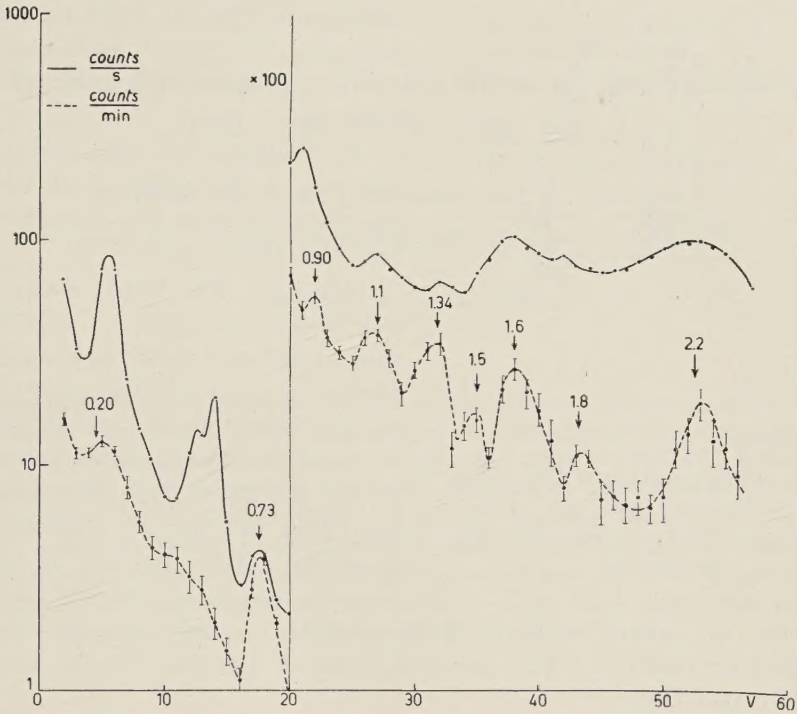

Fig. 2. – γ-ray spectrum of $^{228}_{90}$Th in equilibrium with its decay products (continuous line); γ-ray spectrum of $^{212}_{83}$Bi (dotted line). Energy is indicated through arrows and is in MeV.

The probable errors of the experimental points of the single spectrum are not represented, they are $< 1\%$.

The experimental points of the spectrum in coincidence give the triple effective coincidences corrected for random coincidences.

3. – Discussion.

It will be seen that the coincidence curve shows the following peaks: 0.20, 0.73, 0.90, 1.10, 1.34, 1.50, 1.60, 1.80, 2.20 MeV.

Actually the 0.73, 1.34, 1.50, 1.60, 1.80 and 2.20 MeV γ-rays which in the previous research appeared to be consistent with the results obtained by various authors, exist without uncertainty.

In the range of low energies we observe a γ-ray having energy of 0.20 MeV. No peak at 0.83 MeV appears in the coincidence spectrum (*). Therefore we think that the γ-ray of this energy, reported by JOHANSSON must not be ascribed to $^{212}_{83}$Bi. It could be considered belonging to $^{208}_{81}$Tl; indeed this nuclide has a γ transition having energy of 0.860 MeV.

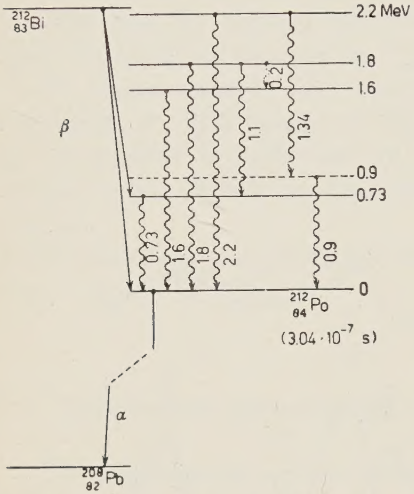

Fig. 3. – Decay scheme of $^{212}_{83}$Bi. Long range α-particles, are not shown. Dotted level and ray are tentative assigments.

Furthermore we can deduce from our results the existence of 0.90 (ITOH and WATASE found a 0.96 MeV γ-ray) and 1.10 MeV γ-rays.

Although this latter could be the pair production peak of 1.60 MeV representing escape of one annihilation photon, a γ-ray of this energy belongs to $^{212}_{83}$Bi according to CURRAN *et al.*

Moreover taking into account the levels originated by the long range α-particles and the γ-γ cascades experimentally verified by one of us ([4]) we suggest the decay scheme of Fig. 3. However we cannot establish if the 0.90 MeV γ-ray is followed by the 1.34 MeV or viceversa. The proposed level at 0.90 MeV should be verified by the β-spectrum.

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The authors wish to thank Prof. E. PERUCCA for his kind interest in this work.

(*) This result agrees with that of J. BURDE and B. ROZNER (*Phys. Rev.*, **107** II, 531 (1957)) which appeared while this paper was already submitted to publication.

([4]) F. DEMICHELIS: *Nuovo Cimento*, **12**, 407 (1954).

RIASSUNTO

Le coincidenze discriminate tra i raggi γ del $^{212}_{83}$Bi e le particelle α del $^{212}_{84}$Po permettono di separare i raggi γ del nuclide in studio, avente periodo di dimezzamento estremamente breve ($3.04\cdot10^{-7}$ s), da quelli emessi dagli altri nuclidi presenti nella sorgente di $^{228}_{90}$Th in equilibrio. I risultati ottenuti con tale metodo eliminano le incertezze che ancora rimanevano nelle nostre precedenti ricerche e permettono di dare uno schema di decadimento del $^{212}_{83}$Bi.

Nuclear Spectroscopic Investigations of the Nuclides $^{228}_{90}Th$, $^{224}_{88}Ra$, $^{212}_{82}Pb$, $^{212}_{83}Bi$, $^{208}_{81}Tl$.

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(ricevuto il 9 Luglio 1957)

Summary. — The γ spectrum of $^{228}_{90}Th$ in equilibrium with its decay products has been investigated by means of two scintillation spectrometers and a fast slow coincidence set up in the range (0.05 ÷ 0.76) MeV. From our results we may confirm the decay schemes of the nuclides $^{212}_{82}Pb$ and $^{228}_{90}Th$. For the nuclides $^{224}_{88}Ra$ and $^{212}_{83}Bi$ we ascertain the existence of two γ-rays, which fit the energy levels already proposed by other Authors, through a direct measurement. Moreover we find some till now unknown γ-rays which must be assigned to the $^{208}_{81}Tl \rightarrow ^{208}_{82}Pb$ decay. According to these data we propose a new decay scheme for $^{208}_{81}Tl$.

1. – Introduction.

The research reported in this paper is concerned with the investigation of the disintegration schemes of the above nuclides of the thorium series.

In our experiment the radioactive source contains all these nuclides in equilibrium. Therefore the γ-spectrum is rather complex, and it is most useful to use a fast-slow coincidence method, in order to select from the γ-spectrum only the energies of the γ-rays whose coincidence must be recorded.

We have improved our conventional coincidence apparatus (1) by adding two single channel pulse height analyzers which perform the above selection.

Owing to these advantages, a reexamination of the γ-spectra seemed us useful.

(1) F. Demichelis, R. A. Ricci and G. Trivero: *Nuovo Cimento*, 3, 377 (1956).

In conclusion our research gets a direct control of already proposed data, as reported in the Table of Isotopes by HOLLANDER, PERLMAN and SEABORG (2) and in the *New Nuclear Data* of the United States Atomic Energy Commission, and it has given us some new results.

2. – Experimental apparatus.

Fig. 1 shows the block-diagram of the experimental set-up. The source T is ${}^{228}_{90}$Th in equilibrium with its decay products. Its activity was rather weak: $\approx 35\ \mu$c.

Detectors D_1 and D_2 consist of two photomultiplier tubes (DuMont 6292) with attached NaI(Tl) crystals ($\frac{3}{4}$ in. $\times \frac{3}{4}$ in.)

Some measurements were performed by replacing one of the above crystals with a 2 in. $\times$ 2 in. crystal.

In order to stop completely β-particles from the source, a lead shield Pb (1 mm thick) was interposed between source and detectors. The same shield reduces the Compton backscattered photons; indeed the photons backscattered from one detector can reach the other one, producing false coincidences.

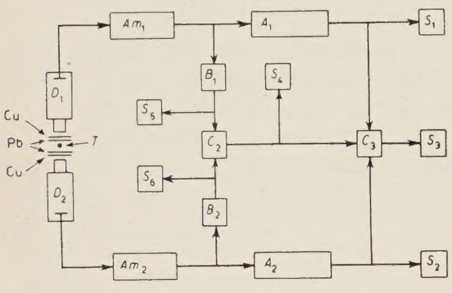

Fig. 1. – Block-diagram of the experimental apparatus. D_1, D_2: NaI(Tl) detectors and 6292 DuMont photomultiplier tubes. S_1, S_2, S_3, S_4, S_5, S_6: scalers. C_2: fast coincidence circuit ($\tau_2 = 1.95 \cdot 10^{-7}$ s). C_3: triple coincidence circuit ($\tau_3 = 3.08 \cdot 10^{-6}$ s). Am_1, Am_2: linear amplifiers. A_1, A_2: single channel pulse height analyzer. B_1, B_2: pulse height discriminators.

This antiscattering lead shield is faced with a graded shield Cu (≈ 0.3 mm Cu) to prevent X-rays generated in the Pb plate from reaching the crystal.

In the range of very low energies (80 keV) the measurements were carried out using either the lead shield Pb or a beryllium shield in order to make sure that the peak at 80 keV was coming from the source and not from scattering from the lead covering the crystals.

The pulses from the two detectors were fed through the linear amplifiers Am_1 and Am_2 to the fast coincidence circuit C_2 (resolving time $\tau_2 = 1.95 \cdot 10^{-7}$ s) and, at the same time, to the single channel analyzers A_1 and A_2.

Pulses coming from C_2, A_1 and A_2 where fed to the triple coincidence cir-

(2) J. M. HOLLANDER, I. PERLMAN and G. T. SEABORG: *Rev. Mod. Phys.*, **25**, 469 (1953).

cuit C_3. In this manner only the pulses which are coincident within the time τ_2 in C_2 and which have heights selected by A_1 and A_2, are recorded at the output of C_3.

The resolving time of this latter is large enough ($\tau_3 = 3.08 \cdot 10^{-6}$ s) in order to account for the delays in the two channels for any setting of the pulse height dials of the analysers.

The analyser A_2 had the window set on the peak of a particular γ-ray (leading radiation), and the γ-spectrum in coincidence with such leading γ-ray was scanned with the various positions of the window of the analyser A_1 along the energy axis.

Scalers S_1, S_2, S_5 and S_6 give us the separate counting rates of the pulses in order to obtain the accidental coincidences.

The usual corrections have been made in order to take into account the fluctuations in the efficiences of the detectors in the course of time.

The energy calibration was obtained with the γ-rays of $^{137}_{55}$Cs, $^{60}_{27}$Co and the 2.615 MeV γ line of the $^{208}_{81}$Tl.

3. – Coincidence results.

Fig. 2 shows a « total » γ spectrum of $^{228}_{90}$Th in equilibrium with its decay products in the range of energies $(0.05 \div 0.6)$ MeV, taken with the single channel analyzer A_1. The statistical errors are not reported; they are evaluated $< 1\%$.

In this spectrum, as well as in the following ones, energies are indicated through vertical arrows, and are in MeV.

Fig. 3 shows the spectrum of the γ-rays in coincidence with the γ radiation of 2.615 MeV of $^{208}_{81}$Tl. As said in Sect. 2, this spectrum was obtained with the window of the analyser A_2 set on the photoelectric peak of the 2.615 MeV γ-ray, and scanning with the analyser A_2 (window 0.5 V) the γ-spectrum from 0.05 MeV to 0.76 MeV.

Owing to the large energy of the leading radiation, in this set of measurements the detector D_2 was the 2 in. $\times$ 2 in. crystal.

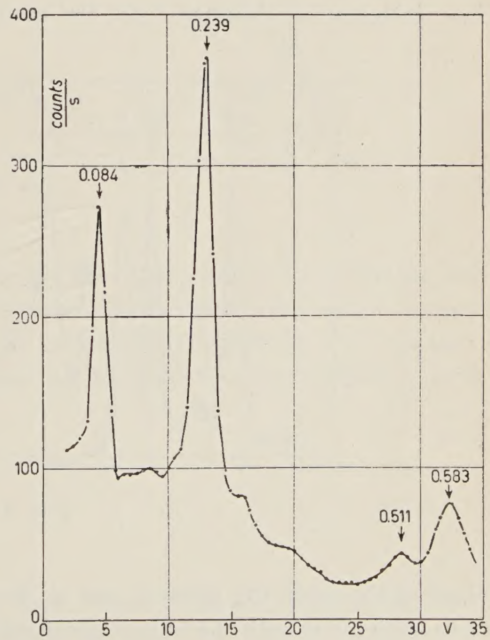

Fig. 2.

Fig. 4, 5, 6 show the spectra of the γ-rays in coincidence with the 0.239 MeV of $^{212}_{82}$Pb, 0.277 MeV of $^{208}_{81}$Tl and 0.084 MeV of $^{228}_{90}$Th respectively.

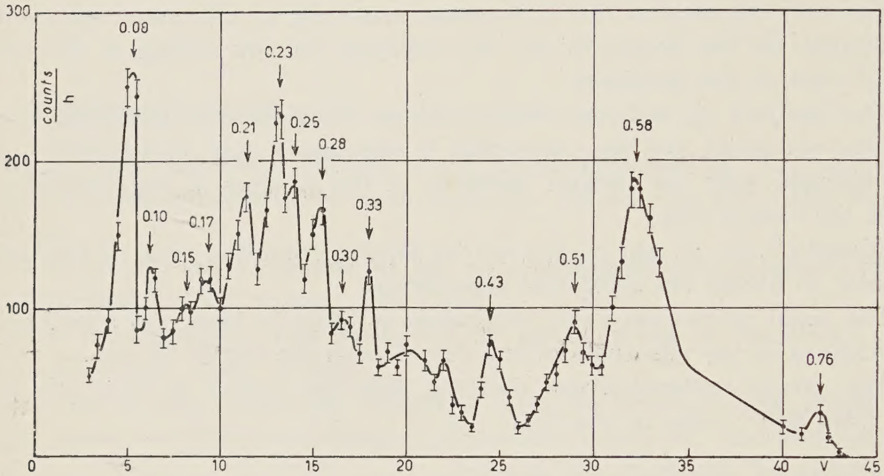

Fig. 3

In Fig. 3, 4, 5, 6 the experimental points with their probable errors give the triple effective coincidences corrected for the accidental coincidences. These latter were obtained from the counting rates of various scalers and the

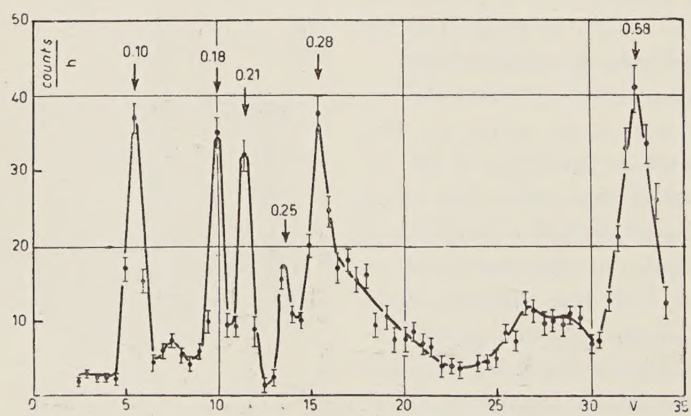

Fig. 4.

values of the resolving times τ_2 and τ_3; the accidental coincidences so evaluated are in agreement with the values obtained from measurements of non-coincident radiations incident on the two detectors from separate sources.

In order to account for a possible shift of the spectrum along the energy axis due to changes of thresholds in the analysers or to changes of the high voltage power supply, etc., before each measurement corresponding to each

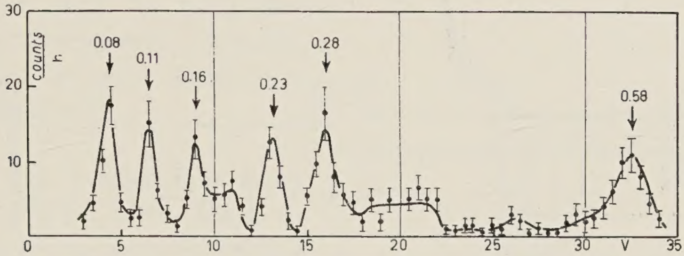

Fig. 5.

experimental point the apparatus was checked verifying that the position of the 0.239 MeV photoelectric peak in the spectrum remained unchanged, and, if necessary, it was readjusted.

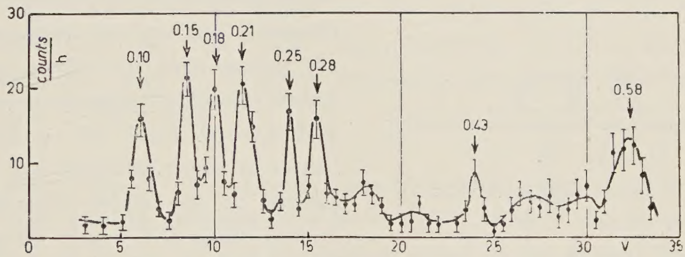

Fig. 6.

In this manner we were allowed to get the final results by summing the statistical material of many sets of measurements with consequent improvement of the accuracy. As a matter of fact for each spectrum the measurements were repeated at least three times, and for a certain part of the spectra even ten times.

4. – Discussion.

From the experimental data of Fig. 3, 4, 5 and 6 we can deduce the existence of γ-rays coincident respectively with the 2.615 MeV, 0.239 MeV, 0.277 MeV and 0.084 MeV γ-rays.

As we used a source of ${}^{228}_{90}$Th in equilibrium with its decay products, the problem arises of assigning the γ-rays to the various nuclides present in the source.

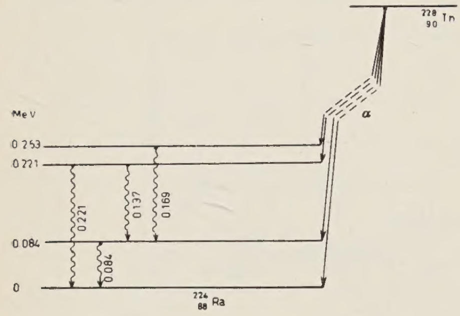

Fig. 7.

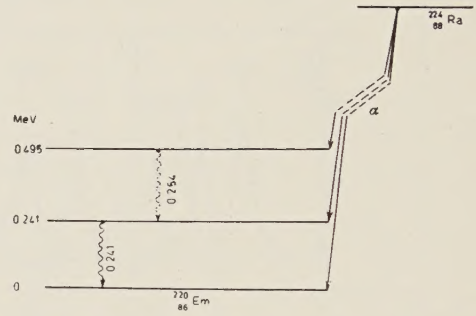

Fig. 8.

The nuclides whose γ-rays are involved in our measurements are listed below. In the decay schemes of these nuclides, shown in Fig. 7, 8, 9, 10, 11, continuous horizontal lines represent already known levels; dotted horizontal lines represent levels obtained from our research and not yet reported; γ-rays are represented by waved lines: continuous for already known γ-rays, dotted and dashed lines for γ-rays already found by other authors, but not yet arranged in the decay scheme, simply dotted lines for γ-rays obtained from our research and not yet reported.

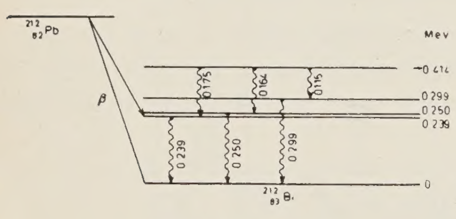

Fig. 9.

${}^{228}_{90}$Th. – The levels are known from the α emission; moreover the following γ-rays exist having energies 0.084 MeV, 0.137 MeV, 0.169 MeV and 0.212 MeV respectively; these rays could originate the proposed decay scheme of Fig. 7 (3).

According to Fig. 6 the 0.15 MeV and the 0.18 MeV γ-rays are coincident with the 0.084 MeV γ-ray.

These rays may be assigned either to the decay ${}^{228}_{90}$Th → ${}^{224}_{88}$Ra, or to the decay ${}^{208}_{81}$Tl → ${}^{208}_{82}$Pb (see Fig. 11).

However, taking into account all the experimental data, we think that the scheme of Fig. 7 could be considered verified.

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(3) F. Asaro, F. Stephens and I. Perlman: *Phys. Rev.*, **92**, 1495 (1953); C. J. D. Jarvis: *Proc. Phys. Soc.*, A **66**, 1074 (1953); F. Stephens, F. Asaro and I. Perlman: *Phys. Rev.*, **96**, 1568 (1954).

$^{224}_{88}$Ra. – The levels, as indicated by the α spectrum (4), appear in Fig. 8. The 0.241 MeV γ-ray is a transition from the first excited state. From Fig. 4 we could deduce a coincidence between a 0.25 MeV γ-ray and 0.239 MeV γ-ray which can be assigned to $^{224}_{88}$Ra. On the other hand the 0.254 MeV γ-ray has a low intensity (≈ 0.4%) and therefore the peak observed is not very prominent.

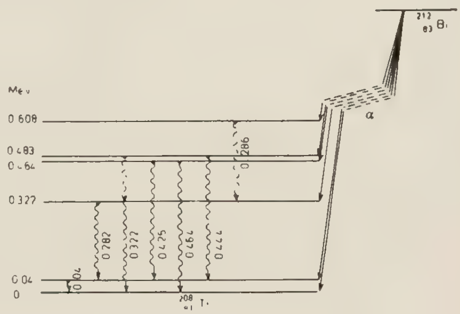

Fig. 10.

$^{212}_{82}$Pb. – Fig. 9 shows the decay scheme of $^{212}_{82}$Pb (5). From Fig. 4 we may deduce a coincidence between a 0.239 MeV and a 0.18 MeV γ-ray. From Fig. 5 we may deduce a coincidence between a 0.299 MeV and a 0.11 MeV γ-ray.

On the other hand we cannot exclude that the same coincidence deduced from Fig. 4 may be ascribed to the 0.250 MeV and to the 0.164 MeV γ-rays in Fig. 9.

Hence, while we have no doubt in the assignment of the γ-γ cascade (0.115 ÷ 0.299) MeV to the $^{212}_{82}$Pb, the cascades (0.175 ÷ 0.239) MeV or (0.169 ÷ 0.250) MeV can be assigned either to the decay $^{212}_{82}\text{Pb} \to {}^{212}_{83}\text{Bi}$ or to the decay $^{208}_{81}\text{Tl} \to {}^{208}_{82}\text{Pb}$ (see Fig. 11).

$^{212}_{83}$Bi. – From Fig. 5 we deduce a coincidence between a 0.277 MeV γ-ray and two γ-rays having energies 0.16 MeV and 0.28 MeV respectively.

While the 0.16 MeV γ-ray can be assigned either to the decay $^{212}_{83}\text{Bi} \to {}^{208}_{81}\text{Tl}$ (6) (see Fig. 10) or to the decay $^{208}_{81}\text{Tl} \to {}^{208}_{82}\text{Pb}$ (see Fig. 11), the 0.286 MeV γ-ray is certainly a transition in the decay $^{212}_{83}\text{Bi} \to {}^{208}_{81}\text{Tl}$. We can deduce therefore the existence of a 0.284 MeV γ-ray coincident with a 0.282 MeV γ-ray in the decay $^{212}_{83}\text{Bi} \to {}^{208}_{81}\text{Tl}$.

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(4) S. Rosenblum, M. Valadares and M. Guillot: *Compt. Rend.*, **234**, 1767 (1952); D. E. Muller, H. C. Hoyt, D. J. Klein and J. W. M. DuMond: *Phys. Rev.*, **88**, 775 (1952); F. Asaro, F. Stephens and I. Perlman: *Phys. Rev.*, **92**, 1495 (1953); S. Rosenblum, M. Valadares and M. Guillot: *Journ. Phys. et Rad.*, **15**, 129 (1954).

(5) J. Surugue: *Journ. Phys. et Rad.*, **7**, 145 (1946); N. Feather, J. Kyles and R. W. Pringle: *Proc. Phys. Soc.*, **61**, 466 (1948); D. G. E. Martin and H. O. W. Richardson: *Proc. Roy. Soc.*, A **195**, 287 (1948).

(6) B. B. Kinsey: *Phys. Rev.*, **72**, 526 (1947); A. Rytz: *Journ. Recherches Centre Nat. Recherche Sci. Labs. Bellevue*, no. 25, 254 (1953); I. Perlman and F. Asaro: *Ann. Rev. of Nucl. Sci.*, **4**, 157 (1954); G. H. Briggs: *Rev. Mod. Phys.*, **26**, 1 (1954); O. B. Nielsen: *Dan. Mat. Fys. Medd.*, **30**, no. 11 (1955).

$^{208}_{81}$Tl. – Fig. 11 shows the decay scheme of $^{208}_{81}$Tl.

From Fig. 3 the 2.615 MeV γ-ray of $^{208}_{81}$Tl is coincident with the new following γ-rays: 0.08, 0.10, 0.15, 0.21, 0.25, 0.30, 0.33, 0.43, 0.76 MeV.

The 0.25 MeV and 0.76 MeV γ-rays, according to ELLIOTT *et al.*, come from a level at 3.961 MeV not shown in Fig. 11.

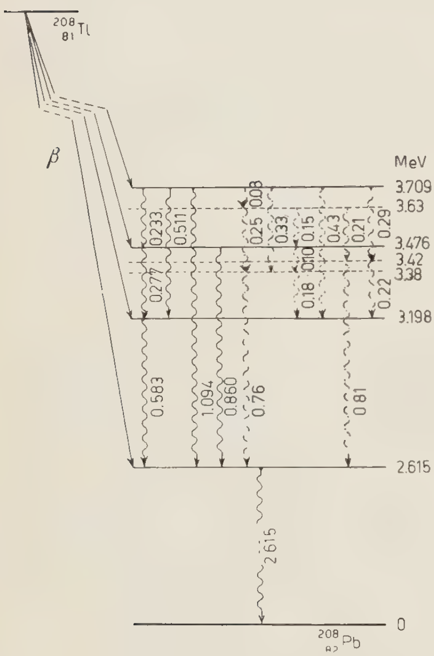

Fig. 11.

From Fig. 4 the 0.233 MeV γ-ray of $^{208}_{81}$Tl is coincident with the following γ-rays which can be assigned to $^{208}_{81}$Tl: 0.10, 0.18, 0.21, 0.28, 0.58 MeV.

From Fig. 5 the 0.277 MeV γ-ray of $^{208}_{81}$Tl is coincident with the following γ-rays which can be assigned to $^{208}_{81}$Tl: 0.08, 0.16, 0.23, 0.58 MeV.

On the other hand, according to SIEGBAHN and GERHOLM (7), the γ-rays 0.811, 0.618, 0.330, 0.303, 0.252, 0.211, 0.147 or 0.075 MeV must be assigned to $^{208}_{81}$Tl.

Our experimental results are consistent with the existence of a 0.08 MeV γ-ray. This ray cannot come from the lead shield as said above; therefore the peak at 0.08 MeV corresponds to a ray coming from the source.

Actually the γ-rays of $^{208}_{81}$Tl give rise to Pb X-rays following internal conversion electrons. The prominent peak corresponding to 0.08 MeV γ or X-rays in this case should be coincident with all the γ-rays of $^{208}_{81}$Tl. On the contrary in Fig. 6 the 0.08 MeV radiation is not coincident, for instance, with the 0.233 MeV γ-ray, while it appears coincident with other rays of $^{208}_{81}$Tl which are not present in the proposed decay scheme (8).

Moreover since the 0.252 MeV and the 0.763 MeV γ-rays obtaind by ELLIOTT *et al.* and evidenced by our experimental results are both coincident with the 2.62 MeV and 0.08 MeV γ-rays, and the difference in energy between the 3.709 MeV and 2.615 MeV levels is just the sum of the energies 0.763,

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(7) K. SIEGBAHN and T. R. GERHOLM: *Beta and Gamma-ray Spectroscopy* (Amsterdam 1955), p. 920.

(8) L. G. ELLIOTT, R. L. GRAHAM, J. WALKER and J. L. WOLFSON: *Phys. Rev.*, **93**, 356 (1954); **94**, 795 A (1954); *Proc. Roy. Soc. Can.*, **48**, 12 A (1954).

0.252, 0.08 MeV, we think that the existence of a cascade (0.08—0.252—0.763) MeV is most probable.

In conclusion, as a result of the above argument, we propose the decay scheme of Fig. 11 for the nuclide $^{208}_{81}$Tl.

In the preceding discussion are then shown the disintegration schemes of the nuclides under investigation.

In these schemes we have reported not only the data which result from our research but also the previous ones.

As said above in the energy schemes are shown in a different manner those levels and γ-rays proposed by us and those which have been subjected to a direct verification with the coincidence method described.

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RIASSUNTO

Mediante il metodo delle coincidenze discriminate si è studiato lo spettro γ del $^{228}_{90}$Th in equilibrio con i suoi prodotti di decadimento, nel campo di energie $(0.05 \div 0.76)$ MeV. Dai risultati ottenuti si ha la conferma dello schema di decadimento del $^{212}_{82}$Pb e del $^{228}_{90}$Th. Per i nuclidi $^{224}_{88}$Ra e $^{212}_{83}$Bi accertiamo l'esistenza di due raggi γ che si inquadrano negli schemi di decadimento già proposti da altri Autori. Inoltre si deduce l'esistenza di alcuni nuovi raggi γ da attribuire al decadimento $^{208}_{81}\mathrm{Tl} \rightarrow {}^{208}_{82}\mathrm{Pb}$, sicchè si propone uno schema di decadimento.

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Clothed Particle Operators in Simple Models of Quantum Field Theory.

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Summary. — The idea of using clothed operators in the quantum theory of fields, i.e. operators which create and annihilate one particle eigenstates of the total Hamiltonian, is examined. In cut-off theories, these clothed operators are related to the unrenormalized Schrödinger picture operators by a similarity transformation. In terms of these clothed operators the virtual cloud of particles surrounding a physical particle appearing in present discussions of field theory no longer appears, corresponding to the elimination, in the terminology of Van Hove, of persistent effects. Mass and wave function renormalization are automatically accomplished by unitary clothing transformations, and the usual renormalization program is extended to the renormalization of the states appearing in the theory. The form of the Hamiltonian expressed in terms of clothed operators is discussed and these ideas are applied to the static scalar model, the Lee model, and the Ruijgrok-Van Hove model.

Introduction.

Recent formulations of the quantum theory of fields (1) have tended to avoid the Hamiltonian formalism. These versions base the theory on certain general assumptions such as the principle of microscopic causality (local commutativity of field operators), asymptotic conditions, Lorentz invariance, and on the properties of the eigenstates of the energy-momentum four vector of the system to derive analytic properties of the amplitudes which are to describe

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(1) See in particular, H. Lehmann, K. Symanzik and W. Zimmermann: *Nuovo Cimento*, 6, 319 (1957) (LSZ).

scattering processes (dispersion relations). Such theories do not make any specific dynamical assumptions and are characterized only by an S-matrix. Since their aim is to discover whether a divergence-free and consistent relativistic description of the interaction of elementary particles based on the aforementioned assumptions exists, LSZ have not discussed whether these formulations can be based on a Lagrangian or Hamiltonian action principle.

An S-matrix formulation avoids one of the basic difficulties encountered in the usual Lagrangian or Hamiltonian formulation of field theory, namely, that by virtue of the interaction each quantum (which is defined in terms of the unperturbed Hamiltonian of the field) acquires a persistent cloud of other quanta around it. Furthermore, in the limit of point interactions, these cloud effects give rise to divergences. The renormalization program in part circumvented these difficulties by noting that by virtue of the interaction the constants appearing in the theory, such as the mass of the bare quanta and the coupling constant, are altered and cannot be identified with the corresponding measured quantities. As is well known, the redefinition of coupling constant and mass parameter proved sufficient in quantum electrodynamics to circumvent all divergence difficulties in the S-matrix. Nonetheless as has been emphasized recently by VAN HOVE (2), the current renormalization program is still unsatisfactory in that the states and operators in terms of which the theory is formulated are essentially « bare » states and « bare » operators describing particles without their clouds and defined in terms of the unobservable unperturbed Hamiltonian.

VAN HOVE, in the papers (2) referred to, has made a penetrating analysis of these cloud effects. These papers were in fact the starting point of the present investigation. The essential achievement of the analysis by VAN HOVE, besides clarifying further the mathematical structure of theories with cloud effects, is an explicit determination of the perturbed stationary states which exhibit in a clear fashion the attached or cloud states. As VAN HOVE points out, an unsatisfactory feature of this analysis is the fact that the results depend so explicitly on the particular breakup of the Hamiltonian into a perturbed and unperturbed part.

In the present paper we investigate these cloud effects in a manner which is more closely related to the usual field theoretic procedure than Van Hove's analysis. Our aim is to formulate the theory in such a way that the states and operators appearing therein refer only to physical or « clothed » particles (*). Our point of departure is the usual description of field theory based upon a Hamiltonian $H=H_0+H_I$, where H_0 describes the « bare » quanta of the fields and H_I represents a perturbation which in addition to the mutual interaction

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(2) L. VAN HOVE: *Physica*, **21**, 901 (1955); **22**, 343 (1956).

(*) Compare: G. C. WICK: *Rev. Mod. Phys.*, **27**, 339 (1955).

between quanta also causes self interactions which give rise to the aforementioned cloud effects. We then analyse the physical vacuum and one particle states of the theory. The former is an eigenfunction of the total Hamiltonian having zero energy and zero momentum, and which, in Lorentz invariant theories, is invariant under arbitrary Lorentz transformations. It corresponds to a state in which there are no physical particles present. The physical one particle states are eigenfunctions of the total Hamiltonian and of the total momentum operator for which the energy and momentum eigenvalues are related in the form usually ascribed to a free physical particle. For a relativistic theory this relation is of the form $E_p = \sqrt{m^2 + p^2}$. In our context the parameter m is the mass of the physical object identified with the « particle » of the theory. In addition, these states must be eigenfunctions of whatever observables further characterize a one particle state. Thus in quantum electrodynamics they must be eigenfunctions of the total charge: for the positon or negaton state this will be $\pm e$, where e is the measured electronic charge. These one particle states can be expressed in terms of eigenfunctions of H_0 as a superposition of such unperturbed states admixed to the « bare » one quantum state. These attached states are the ones usually referred to as constituting the cloud.

We now fulfil our requirement that the theory be expressed in terms of clothed particle states by introducing operators $\psi_i^\dagger$ with the property that, operating on the physical vacuum $\Psi_0 = |0\rangle_c$, they create the correct one particle states; more precisely if

$$H|0\rangle_c = 0\,, \tag{1}$$

we require that

$$\psi_i^\dagger(p, E_i, m_i)|0\rangle_c = |p, E_i, m_i\rangle_c \tag{2}$$

and

$$H\psi_i^\dagger(p, E_i, m_i)|0\rangle_c = E_i\psi^\dagger(p, E_i, m_i)|0\rangle_c \tag{3}$$

$$P\psi_i^\dagger(p, E_i, m_i)|0\rangle_c = p\psi^\dagger(p, E_i, m_i)|0\rangle_c \tag{4}$$

$$E_i = E_i(p_i, m_i)\,, \tag{5}$$

where (5) is the energy-momentum relation appropriate to the particular particle under consideration. We also introduce the corresponding annihilation operators $\psi_i(p, E_i, m_i)$ which destroy the vacuum

$$\psi_i(p, E_i, m_i)|0\rangle_c = 0 \tag{6}$$

and which transform the physical one particle states into the vacuum

$$\psi_i(p, E_i, m_i)|p', E_i', m_i\rangle_c = \delta(p - p')|0\rangle_c\,. \tag{7}$$

We may define these clothed operators in terms of the bare operators by introducing a transformation W, such that if $\psi_i(p, E_i^{(0)}, m_i^{(0)})$ is the bare quantum operator of type i, energy momentum $E_i^{(0)}$, p bare mass $m_i^{(0)}$,

$$\boldsymbol{\psi}_i(p, E_i, m_i) = W\psi(p, E_i^{(0)}, m_i^{(0)})W^{-1}\,, \tag{8a}$$

$$\boldsymbol{\psi}_i^\dagger(p, E_i, m_i) = W\psi^*(p, E_i^{(0)}, m_i^{(0)})W^{-1}\,. \tag{8b}$$

The above requirements on $\boldsymbol{\psi}_i$ are then guaranteed if

$$W|0\rangle = |0\rangle \tag{9}$$

$$\begin{aligned} W|p, E_i^{(0)}, m_i^{(0)}\rangle &= W\psi^*(p, E_i^{(0)}, m_i^{(0)})W^{-1}W|0\rangle \\ &= |p, E_i, m_i\rangle_c \end{aligned} \tag{10}$$

where

$$H_0|0\rangle = 0 \tag{11}$$

$$H_0|p, E_i^{(0)}, m_i^{(0)}\rangle = E_i^{(0)}|p, E_i^{(0)}, m_i^{(0)}\rangle\,. \tag{12}$$

We call such W transformations clothing transformations. The clothing transformation is not uniquely determined by the above requirements. However, the qualitative properties of the theory when expressed in terms of clothed operators, such as the absence of persistent effects discussed in Sect. **1**, do not depend on which clothing transformation is used. In this paper we are not concerned with this lack of uniqueness. It corresponds to alternative and equivalent descriptions of the interaction between two and more particles. For theories with finite rest mass quanta the requirement that when two or more particles are separated by large spatial distances the total energy and momentum of the states be the sum of the energies and momenta of the individual particles seems to be automatically satisfied in our Hamiltonian formulation, so that this requirement is not a separate restriction on the clothing transformation. It is likely that the coupling constant renormalization, which can be taken as a requirement on a given two particle state (e.g. that the scattering cross-section between the two particles take on a given value in the limit of zero energy) will further restrict the clothing transformation.

We call the operator $\boldsymbol{\psi}_i$ a clothed particle operator, since it creates eigenstates of the total Hamiltonian. It should be noted that the clothed operators do not correspond to the usual renormalized Heisenberg operators since the latter do not have this property. This fact is evident from the LEHMANN [3] canonical form for the twofold vacuum expectation value of—say—the Hei-

[3] H. LEHMANN: *Nuovo Cimento*, **2**, 342 (1954).

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senberg operator $\varphi(x_\nu)$ for a neutral meson field

$$\langle\Psi_0|\varphi(x_\nu)\varphi(y_\nu)|\Psi_0\rangle = i\int \mathrm{d}\mu^2\varrho(\mu^2)\,\Delta^{(+)}(x_\nu - y_\nu, \mu^2)\ , \tag{13}$$

where the weight function $\varrho(\mu^2)$ is a positive measure characterizing the particular theory. If $\varphi(x_\mu)\Psi_0$ were an eigenfunction of the total Hamiltonian corresponding to a physical meson of observed mass m (or more exactly an eigenfunction of $P_\mu P^\mu$ with eigenvalue m^2) then necessarily $\varrho(\mu^2) = c\,\delta(\mu^2 - m^2)$ where we could choose $c = 1$ by proper normalization. However, in all present theories which contain non-trivial interactions, $\varrho(\mu^2) = \delta(\mu^2 - m^2) + \varrho'(\mu^2)$, where $\varrho'(\mu^2) = 0$ for $\mu^2 \leqslant m^2$, but ϱ' does not vanish everywhere. For those values of μ^2 for which $\varrho' \neq 0$, there are states of mass $\mu^2 \neq m^2$ in $\varphi(x_\nu)\Psi_0$. Furthermore the departure of ϱ' from zero is indicated by the departure of the meson field strength renormalization Z_3 from unity since,

$$Z_3^{-1} = \int_0^\infty \varrho(\mu^2)\mathrm{d}\mu^2 = 1 + \int_{m^2}^\infty \varrho'(\mu^2)\,\mathrm{d}\mu^2\ . \tag{14}$$

In the present formulation, as in current renormalization theory, the one particle states acquire a certain privileged position compared to other states of the system. This is in line with the view that they represent—so to say—the building blocks of the theory. In fact, an exact determination of W will be obtained only if we can exhibit in closed form the one particle states. The present paper concerns itself primarily with certain simple theories for which the one particle states can be solved exactly. Thus, after a general discussion of clothing transformations in Sect. **1**, Sect. **2** considers the simple scalar field, Sect. **3** the Lee model, and Sect. **4** the Ruijgrok-Van Hove model. Sect. **5** discusses some general consequences of the viewpoint adopted and gives a preliminary estimate of the situation which may be expected in the relativistic case.

1. – The clothing transformation.

In the present section we discuss the requirements imposed on our clothed operators as well as their general properties. As stated in the introduction we accept the usual Hamiltonian formulation of field theory. As a specific example consider the case of a fermion field $(\psi(p))$ interacting with a Boson field $(a(k))$. We introduce operators $\boldsymbol{\psi}(p)$ and $\boldsymbol{a}(k)$, which acting on the physical vacuum

state $|0\rangle_c$ create physical one particle states

(1) $$\psi^\dagger(p)|0\rangle_c = |p;\rangle_c\,; \qquad H|p;\rangle_c = \varepsilon(p)|p;\rangle_c\,,$$

(2) $$\boldsymbol{a}^\dagger(k)|0\rangle_c = |\,;k\rangle_c\,; \qquad H|\,;k\rangle_c = \omega(k)|\,;k\rangle_c\,,$$

where $\varepsilon(p)$ and $\omega(k)$ are the observed (renormalized) energies of the Fermion and Boson, respectively.

The corresponding annihilation operators $\psi(p)$, $\boldsymbol{a}(p)$ which destroy the physical vacuum and which transform the physical one particle state into the vacuum have the property:

(3a) $$\psi(p)|p';\rangle_c = \delta(p-p')|0\rangle_c\,,$$

(3b) $$\boldsymbol{a}(k)|\,;k'\rangle_c = \delta(k-k')|0\rangle_c\,.$$

We define these clothed operators by a similarity transformation (hereafter called clothing transformation) on the bare operators

(4) $$\psi(p) = W\psi(p)W^{-1}\,; \qquad \psi^\dagger(p) = W\psi^*(p)W^{-1}\,,$$

(5) $$\boldsymbol{a}(k) = Wa(k)W^{-1}\,; \qquad \boldsymbol{a}^\dagger(k) = Wa^*(k)W^{-1}\,,$$

which has the consequence that the clothed operators $\psi^\dagger$, ψ satisfy the same commutation rules as did ψ^*, ψ. These similarity transformations may be generated by both unitary and non-unitary operators W (*). In the case of unitary operators it follows that $\psi^\dagger = \psi^*$. For the remainder of this section we deal with the unitary case. It is clear that the above requirements will be satisfied if $W = U$, $U^* = U^{-1}$

(6) $$U|0\rangle = |0\rangle_c\,,$$

(7) $$U|p;\rangle = U\psi^*(p)|0\rangle = U\psi^*(p)U^{-1}U|0\rangle = \psi(p)|0\rangle_c = |p;\rangle_c\,,$$

(8) $$U|\,;k\rangle = Ua^*(k)|0\rangle = Ua^*(k)U^{-1}U|0\rangle = \boldsymbol{a}^*(k)|0\rangle_c = |\,;k\rangle_c\,.$$

Once the clothed operators are defined they are introduced into the theory by expressing the original Hamiltonian in terms of them. When this substitution is carried out, we obtain a Hamiltonian which is numerically the

(*) In this paper we are concerned only with theories in which cut-off functions are inserted in the interaction Hamiltonian to prevent the occurrence of divergent integrals. The above statement that the canonical transformations are unitary or non-unitary refers to theories which have been made finite using such cut-off functions. In relativistic theories it is likely (4) that the transformation will not be unitary.

(4) R. HAAG: *Danske Vidensk. Selskab*, **29**, 12 (1955); D. HALL and A. S. WIGHTMAN: *Danske Vidensk. Selskab*, **31**, 5 (1957).

same Hamiltonian as before, but which has a different functional form than the original Hamiltonian. We express this by writing $H(\psi, a) = H_c(\boldsymbol{\psi}, \boldsymbol{a})$. Our requirement that $\boldsymbol{\psi}$, $\boldsymbol{a}$ be clothed operators leads to interesting conclusions about the form of H_c. Since $H_c|0\rangle_c = 0$, we have from (1),

$$H_c \boldsymbol{\psi}(p)|0\rangle_c = [H_c, \boldsymbol{\psi}(p)]|0\rangle_c = \varepsilon(p)\boldsymbol{\psi}(p)|0\rangle_c \,. \tag{9}$$

Thus H_c contains a term $\int d^3p\,\varepsilon(p)\psi^*(p)\psi(p)$ which has the free field form with the renormalized energy. Similarly H_c contains such a term for the operators describing the anti-fermions (if the theory is a relativistic one) and a similar term for the Boson field. In terms of clothed operators the mass renormalization is accomplished without the addition and subtraction of counter terms customary in current treatments of renormalization ([5]). This is a consequence of the requirement (1) that the one particle states have the correct physical energy associated with that particle. The sum of these free field terms is now the new free (unperturbed) Hamiltonian H_{0c}

$$H_{0c} = \int d^3p\,\varepsilon(p)\boldsymbol{\psi}^*(p)\boldsymbol{\psi}(p) + \int d^3k\,\omega(k)\boldsymbol{a}^*(k)\boldsymbol{a}(k)\,. \tag{10}$$

We call $H_c - H_{0c}$ the new interaction Hamiltonian H_{Ic}. In terms of this new division of the Hamiltonian into free and interacting parts, self-energy and cloud effects no longer occur for theories where all self-energy effects are associated with the vacuum and one particle states. To characterize H_{Ic} we note that

$$H_{Ic}|0\rangle_c = H_{Ic}\, p;\rangle_c = H_{Ic}|;k\rangle_c = 0\,, \tag{11}$$

so that H_{Ic}, in normal ordered form, is a sum of terms each of which has at least two annihilation operators. Since H_{Ic} is Hermitian each term also has at least two creation operators. Thus if the old interaction Hamiltonian H_I in terms of bare operators was of the form $(\psi^*\psi a + \text{c. c.})$ such a term will no longer appear in H_{Ic}. The simplest type of vertex which will occur in H_{Ic} will correspond to Fermion-Boson scattering, i.e. to an energy-momentum conserving direct interaction between a Fermion and a Boson. In addition to this particular direct interaction term in H_{Ic}, there will occur in H_{Ic} other terms which correspond to all the real (energy-momentum conserving) processes which can occur in the theory.

Self-energy and cloud effects occur in the current formulations of field theory when the iteration of the interaction H_I leads to a state identical to

([5]) Compare S. KAMEFUCHI and H. UMEZAWA: *Prog. Theor. Phys.*, **7**, 399 (1952).

the initial state with a coefficient proportional to the quantization volume. This occurs in particular in connection with the vacuum and one particle states. In terms of clothed operators, iterates of H_{Ic} on a given state can again lead to that state, but only as one member of a continuum of possibilities. These statements can be made more succinct by using Van Hove's notion of diagonal or persistent perturbations. Let $|\alpha\rangle$ be eigenfunctions of H_0 with continuum normalization

$$\langle\alpha'|\alpha\rangle = \delta(\alpha-\alpha')\,. \tag{12}$$

Then in the usual formulations of field theories $\langle\alpha'|H_I|\alpha\rangle$ contains no diagonal terms, i.e. no terms with a factor $\delta(\alpha-\alpha')$. However the matrix elements of iterates of H_I may contain such diagonal terms. In particular if A_j is diagonal in the $|\alpha\rangle$ representation one usually has

$$\langle\alpha|H_I A_1 H_I \dots A_n H_I|\alpha'\rangle = \delta(\alpha-\alpha')\,F_1(\alpha) + F_2(\alpha,\alpha')\,, \tag{13}$$

where F_2 has no $\delta(\varepsilon_0(\alpha)-\varepsilon_0(\alpha'))$ term. The term $\delta(\alpha-\alpha')F_1$ is the diagonal part of the matrix element and diagramatically corresponds to those diagrams in which the particles in the state α propagate without interaction with one another and only undergo self-interactions. The occurrence of such F_1 terms, by definition due to « persistent perturbations », has been shown by VAN HOVE to be responsible for cloud and self-energy effects. Using these ideas, our conclusion is that for theories in which self-energy and cloud effects are associated only with the vacuum and one particle states, the introduction of clothed operators into the theory removes cloud effects because the total Hamiltonian then contains an interaction term H_{Ic} which in the basis of eigenvectors of H_{0c} does not produce persistent perturbations.

In a manner similar to the mass renormalization, the wave function (field strength) renormalization is automatically accomplished by using clothed operators generated by unitary transformations. Källén's [6] prescription for the wave function renormalization is

$$_c\langle 0|\psi'(p')|p\rangle_c = \delta(p-p')\,, \tag{14}$$

where the « renormalized » operator $\psi'(p)$ is defined by

$$\psi'(p) = Z_2^{-\frac{1}{2}}\psi(p)\,. \tag{15}$$

However (14) is automatically satisfied for the clothed operator without in-

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[6] G. KÄLLÉN: *Helv. Phys. Acta*, **25**, 417 (1952).

troducting a multiplicative factor, in fact the much stronger statement

$$\psi^*(p)\,|0\rangle_c = |p\rangle_c \tag{16}$$

is true.

We have as yet not treated the coupling constant renormalization as simply as the other renormalization constants within our Hamiltonian framework.

VAN HOVE (7) (*) has shown that for theories with trilinear point interactions (or with cut-off functions which do not decrease rapidly enough) there are no normalizable vectors other than the vacuum, for those theories where the bare vacuum and physical vacuum coincide, which are in the common domain of both H and H_0. We note that after the clothing transformation has been performed this difficulty is explicitly removed since at least the physical vacuum and the one particle state will be in the common domain of H_0 and H_c. The fact that the disjointness of domains of H and H_0 does not occur is to be expected since it is a consequence of the occurence of a virtual cloud of quanta in the physical state and more specifically, the vanishing of the field strength renormalization constant, both of which difficulties have been eliminated by the introduction of clothed operators.

2. – The scalar field.

The simplest example of a field theory on which the above program can be illustrated is that of a neutral scalar field (8,7) interacting with Fermions whose unrenormalized energy m_0 is taken to be independent of momentum. The Hamiltonian describing this system is $H = H_0 + H_I$

$$\left\{\begin{aligned} H_0 &= m_0\int d^3p\,\psi^*(p)\psi(p) + \int d^3k\,\omega(k)a^*(k)a(k)\,; \qquad \omega(k) = \sqrt{k^2+\mu^2}\,,\\ H_I &= \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int d^3p\int\frac{d^3k f(\omega)}{\sqrt{2\omega(k)}}\,\psi^*(p+k)\psi(p)\big(a(k)+a^*(-k)\big)\,, \end{aligned}\right. \tag{1}$$

where $\psi(p)$ and $a(k)$ are the destruction operators for nucleons and mesons, respectively, which satisfy the commutation rules:

$$[\psi(p),\,\psi^*(p')]_+ = \delta(p-p')\,, \tag{2a}$$

$$[a(k),\,a^*(k')] = \delta(k-k')\,. \tag{2b}$$

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(7) L. VAN HOVE: *Physica*, **18**, 145 (1952); *Bull. Acad. Roy. Belg., Cl. Sc.*, **27**, 1055 (1951).

(*) We thank A. S. WIGHTMAN for a helpful discussion concerning this point.

(8) S. TOMONAGA: *Sci. Pap. I.P.C.R.*, **39**, 247 (1941); G. WENTZEL: *Quantum Theory of Fields* (New York, 1949).

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The function $f(\omega)$ describes the extension of the nucleon and plays the role of a cut-off function, assumed to fall off rapidly enough for large ω to make finite all the integrals that occur in the theory. Since the energy of a nucleon does not depend on its momentum the theory does not take into account recoil effects; however, due to the translational invariance of the Hamiltonian, conservation of momentum holds. Due to the absence of pair effects the no particle state, here the vacuum, is the same for the free as for the coupled fields. The one nucleon state, $\Psi_{(1)}(P)$, of momentum P can be expressed in terms of bare operators as follows:

$$\Psi_{(1)} = \sum_{n=0}^{\infty} \int \mathrm{d}^3p \int \mathrm{d}^3k_1 \ldots \int \mathrm{d}^3k_n c^{(n)}(k_1, k_2 \ldots k_n; p; P) \cdot \\ \cdot \frac{1}{\sqrt{n!}} a^*(k_1) \ldots a^*(k_n) \psi^*(p) \Psi_0 , \tag{3a}$$

where the amplitudes $c^{(n)}$ are given by

$$c^{(n)}(k_1, k_2 \ldots k_n; p; P) = (\Psi_0, \psi(p) a(k_1) \ldots a(k_n) \Psi_1) = \\ = \sqrt{Z_2}\, \delta(p + \sum_i^n k_i - P) \frac{(-\lambda)^n}{\sqrt{n!}} \prod_{i=1}^n \frac{f(\omega_i)}{\sqrt{2(2\pi)^3 \omega^3(k_i)}} . \tag{3b}$$

One readily verifies that this one nucleon state is an eigenfunction of H with eigenvalue m

$$m = m_0 - \lambda^2 \Delta = m_0 + \delta m , \tag{4a}$$

$$\Delta = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3k \, |f(\omega)|^2}{2\omega^2(k)} . \tag{4b}$$

The wave function renormalization constant Z_2 is determined from the requirement that $\Psi_{(1)}$ be normalized and is found to be

$$Z_2 = \exp[-\lambda^2 L] , \tag{5a}$$

$$L = \frac{1}{(2\pi)^3} \int \frac{\mathrm{d}^3k \, |f(\omega)|^2}{2\omega^3(k)} . \tag{5b}$$

The unitary operator $U = \exp[iS]$

$$S = \frac{i\lambda}{\sqrt{(2\pi)^3}} \int \mathrm{d}^3x \psi^*(x) \psi(x) \int \frac{\mathrm{d}^3k f(\omega)}{\sqrt{2\omega^3(k)}} [a^*(k) - a(-k)] \exp[-ikx] , \\ = \frac{i\lambda}{(2\pi)^{\frac{3}{2}}} \int \mathrm{d}^3p \int \mathrm{d}^3k \psi^*(p) \psi(p+k) \frac{f(\omega)}{\sqrt{2\omega^3(k)}} [a^*(k) - a(-k)] , \tag{6}$$

has the property that it leaves the vacuum invariant and that operating on the bare one nucleon state $\psi^*(P)\Psi_0$ will yield the properly normalized « clothed » one nucleon state $\Psi_{(1)}(P)$. This unitary operator thus « attaches » to a nucleon located at the spatial point x its appropriate bare meson cloud. The clothed nucleon operator is defined as

$$(7)\qquad \boldsymbol{\psi}^*(p) = \exp[iS]\psi^*(p)\exp[-iS] = \int \mathrm{d}^3q \int \frac{\mathrm{d}^3x}{(2\pi)^3} \exp[-i(q-p)x]\psi^*(q)\cdot$$
$$\cdot \exp\left[-\frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \frac{\mathrm{d}^3k f(\omega)}{\sqrt{2\omega^3(k)}} \exp[-ikx]\,\big(a^*(k)-a(-k)\big)\right].$$

Operating on the (true) vacuum, $\boldsymbol{\psi}^*(p)$ creates an eigenfunction of the total Hamiltonian corresponding to the presence of one nucleon of mass m and momentum p. Similarly the operator we call the clothed meson (creation) operator is defined as

$$(8a)\qquad \boldsymbol{a}^*(k) = \exp[iS]a^*(k)\exp[-iS] =$$
$$= a^*(k) + \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int \mathrm{d}^3p\,\psi^*(p+k)\psi(p)\,\frac{f(\omega)}{\sqrt{2\omega^3(k)}}\,.$$

The bare and clothed one meson states are thus the same:

$$(8b)\qquad \boldsymbol{a}^*(k)\Psi_0 = a^*(k)\Psi_0\,.$$

This is due to the absence of pair processes in the theory. The Hamiltonian when expressed in terms of clothed operators is given by (*)

$$(9)\qquad H_c(\boldsymbol{\psi}, \boldsymbol{a}) = \exp[-iS(\boldsymbol{\psi}, \boldsymbol{a})]H(\boldsymbol{\psi}, \boldsymbol{a})\exp[iS(\boldsymbol{\psi}, \boldsymbol{a})] = H_{0c} + H_{Ic}\,,$$

where

$$(10)\qquad \begin{cases} H_{0c} = m\displaystyle\int \mathrm{d}^3p\,\boldsymbol{\psi}^*(p)\boldsymbol{\psi}(p) + \int \mathrm{d}^3k\,\omega(k)\boldsymbol{a}^*(k)\boldsymbol{a}(k)\,, \\ H_{Ic} = \dfrac{\lambda^2}{(2\pi)^3}\displaystyle\int \mathrm{d}^3q \int \mathrm{d}^3p \int \frac{\mathrm{d}^3k\,|f(\omega)|^2}{2\omega^2(k)}\,\boldsymbol{\psi}^*(p+k)\boldsymbol{\psi}^*(q)\boldsymbol{\psi}(p)\boldsymbol{\psi}(q+k)\,, \end{cases}$$

and m is the renormalized mass given by Eq. (4).

(*) Note that the clothing transformation is the same function of both the clothed and bare operators, since

$$S(\boldsymbol{\psi}, \boldsymbol{a}) = \exp[iS(\psi, a)]S(\psi, a)\exp[-iS(\psi, a)] = S(\psi, a)\,.$$

This statement holds also for non-unitary clothing transformations.

In terms of dressed operators the new interaction Hamiltonian no longer contains any self-interaction and gives rise only to an interaction between pairs of nucleons. Since the commutation rules of clothed and bare operators are the same, a state such a $\psi^*(p_1) \dots \psi^*(p_n)\Psi_0$ is an eigenstate of H_{0c} belonging to an eigenvalue $\sum_{i=1}^{n} m$ and is asymptotically stationary in the sense of Van Hove [2] (*).

It is interesting to consider the Heisenberg picture for the clothed theory. The dressed nucleon Heisenberg operator is

$$\psi(p, t) = \exp[iH_c t]\,\psi(p)\exp[-iH_c t]\,, \tag{11}$$

whence its equation of motion is

$$-i\partial_t\psi(p, t) = \exp[iH_c t][H_c, \psi(p)]\exp[-iH_c t]$$

or

$$(i\partial_t - m)\psi(p, t) = -\frac{\lambda^2}{(2\pi)^3}\int \mathrm{d}^3q \int \frac{\mathrm{d}^3k\,|f(\omega)|^2}{\omega^2(k)}\,\psi^*(q, t)\psi(q+k, t)\psi(p-k, t)\,. \tag{12}$$

The dressed meson operator obeys a source-free Klein-Gordon equation

$$(\Box + \mu^2)\,\varphi(x, t) = 0\,, \tag{13}$$

in agreement with the fact that the mesons do not interact with the sources. The twofold vacuum expectation value for the dressed nucleon operator is readily computed. Since

$$\begin{cases} H_c\Psi_0 = 0\,, \\ H_c\,\psi^*(p)\Psi_0 = m\,\psi^*(p)\Psi_0 \end{cases} \tag{14}$$

it follows that

$$\begin{aligned} &(\Psi_0, \psi(p, t)\psi^*(p', t')\Psi_0) = \\ &= (\Psi_0, \exp[iH_c t]\,\psi(p, 0)\exp[-iH_c(t-t')]\psi^*(p', 0)\exp[-iH_c t']\,\Psi_0) = \\ &= \exp[-im(t-t')]\,\delta(p-p')\,, \end{aligned} \tag{15}$$

which expresses the fact that the nucleon propagates with mass m, and that the dressed nucleon operators are properly normalized.

From our point of view questions such as the distribution of « bare » mesons in a physical nucleon are meaningless, inasmuch as such statements attri-

(*) A more detailed discussion of asymptotic stationary states is given in Sect. **3**.

bute a physical reality to bare nucleons. What were previously considered to be cloud effects now appear as modifications at small distances in the interaction between dressed particles.

3. – The Lee model (*).

The Lee model [9,10,2] describes three interacting fields, Fermion fields V and N (whose spin is ignored) and a Boson field a (+). The unrenormalized energies of the fields are m_{0V}, m_N and $\omega(k)=\sqrt{\mu^2+k^2}$ respectively, the recoil of the V and N fields being ignored. The Hamiltonian is $H=H_0+H_I$

$$H_0 = m_{0V}\int V^*(p)V(p)\,\mathrm{d}^3p + m_N\int \mathrm{d}^3p N^*(p)N(p) + \int \mathrm{d}^3k\,\omega(k)a^*(k)a(k)\,, \tag{1}$$

$$H_I = \frac{\lambda}{(2\pi)^{\frac{3}{2}}}\int\frac{\mathrm{d}^3kf(\omega)}{\sqrt{2\omega(k)}}\int \mathrm{d}^3p\,V^*(p)N(p-k)a(k) + \text{c.c.}\,. \tag{2}$$

Thus the only interaction produced by the coupling of the fields is $V\rightleftharpoons N+\theta$.

The simplicity of the Lee model is manifested by the fact that the bare and physical vacuum, one N-particle and one θ-particle states coincide,

$$|0\rangle = |0\rangle_c\,;\qquad |N_p\rangle = |N_p\rangle_c\,;\qquad |\theta_k\rangle = |\theta_k\rangle_c\,. \tag{3}$$

The only physical one particle state which differs from the corresponding bare state is the V-state [9,10,2] given by

$$|V_p\rangle_c = \frac{1}{\sqrt{1+\lambda^2L}}\left[|V_p\rangle - \lambda\int \mathrm{d}^3k\bar{F}(\omega)\,|N_{p-k},\theta_k\rangle\right], \tag{4}$$

where

$$F(\omega) = \frac{f(\omega)}{\sqrt{2(2\pi)^3\omega(k)}\cdot(m_N+\omega(k)-m_V)}\,, \tag{4a}$$

and

$$L = \int \mathrm{d}^3k\,|F(\omega)|^2\,. \tag{4b}$$

The factor $1+\lambda^2L$ corresponds to Z_2^{-1} of LEE, N^{-2} of KÄLLÉN, and $[N(p)]^{-1}$

(*) We note at the outset that our methods lead to no new insight into the ghost state.

(9) T. D. LEE: *Phys. Rev.*, **95**, 1329 (1954).

(10) G. KÄLLÉN and W. PAULI: *Dan. Mat. Fys. Medd.*, **30**, No. 7 (1955).

(+) The Boson field quanta are called θ-particles.

of VAN HOVE. Notice that the physical V_c-state contains a bare V-particle together with virtual bare N- and θ-particles. The state $|V\rangle_c$ belongs to a different eigenvalue of H than $|V\rangle$ does of H_0,

$$H_0|V_p\rangle = m_{0V}|V_p\rangle\,, \tag{5a}$$

$$H\;|V_p\rangle_c = m_V\;|V_p\rangle_c\,, \tag{5b}$$

where

$$m_V = m_{0V} + \delta m_V = m_{0V} - \lambda^2 \Delta\,,$$

$$\Delta = \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\,\frac{|f(\omega)|^2}{2\omega(m_N+\omega-m_V)}\,. \tag{6}$$

We obtain a unitary clothing transformation, $U=\exp[iS]$, by noting that the form of $|V\rangle_c$ suggests

$$S = g(\lambda)\left[i\int \mathrm{d}^3p\int \mathrm{d}^3k\bar{F}(\omega)N^*(p-k)a^*(k)V(p) + \text{c.c.}\right]. \tag{7}$$

where $g(\lambda)$ is to be determined. Differentiating the assumed equality

$$\langle V_{p'}|e^{iS}|V_p\rangle = \langle V_{p'}|V_p\rangle_c\,, \tag{8}$$

with respect to λ, we find

$$\frac{-\lambda L}{\sqrt{1+\lambda^2 L}}\,g'(\lambda)\,\delta(p-p') = \frac{-\lambda L}{\sqrt{(1+\lambda^2L)^3}}\,\delta(p-p')\,,$$

$$g'(\lambda) = \frac{1}{1+\lambda^2 L}; \qquad g(\lambda) = \frac{\operatorname{arctg}\lambda\sqrt{L}}{\sqrt{L}}\,. \tag{9}$$

A similar calculation based on $\langle N_{p'},\theta_{k'}|\exp[iS]|V_p\rangle = \langle N_{p'},\theta_{k'}|V_p\rangle_c$ leads to the same $g(\lambda)$. The observation that $\exp[iS]|V\rangle$ contains only the states $|V\rangle$ and $|N,\theta\rangle$, and that $S|0\rangle = S|N\rangle = S|\theta\rangle = 0$ completes the demonstration that the S given by (8) generates a clothing transformation. This S is the simplest possible expression in the sense that its terms are products of the fewest number (3) of operators. All other clothing transformations can be written in the form $\exp[iS']\exp[iS]$, where S' is Hermitian and annihilates the physical vacuum and one-particle states. A set of such transformations is discussed later in this section.

The clothed operators can be expressed as infinite series of normal ordered

products of bare operators. The terms in these series of interest to us here are:

$$\boldsymbol{V}^*(p) = \frac{1}{\sqrt{1+\lambda^2 L}}\left[V^*(p) - \lambda\int \mathrm{d}^3k\bar{F}(\omega)N^*(p-k)a^*(k)\right] + \dots\,, \tag{10}$$

$$\boldsymbol{N}^*(p) = N^*(p) + \dots\,, \tag{11}$$

$$\boldsymbol{a}^*(p) = a^*(p) + \dots\,, \tag{12}$$

$$\begin{aligned}\boldsymbol{N}^*(p)\boldsymbol{a}^*(k) &= N^*(p)a^*(k) + \frac{\lambda}{\sqrt{1+\lambda^2 L}}F(\omega)V^*(p+k) + \\ &+ \frac{1}{L}\left(\frac{1}{\sqrt{1+\lambda^2 L}} - 1\right)\int \mathrm{d}^3k' F(\omega)\bar{F}(\omega')N^*(p+k-k')a^*(k') + \dots\,,\end{aligned} \tag{13}$$

where the dots indicate terms which contain at least one destruction operator to the right.

We now express the Hamiltonian in clothed variables (see equation (2.9)). This calculation can be carried out to any order in λ. We have summed exactly certain terms of interest: the bilinear terms which form the new free Hamiltonian H_{0c}, the trilinear terms which vanish exactly, and the quadrilinear term in $\boldsymbol{N}$, $\boldsymbol{a}$ which represent $\boldsymbol{N}$-$\boldsymbol{\theta}$ scattering. In addition there is an infinite series of other terms (indicated by dots below) which correspond to all real physical processes in the model. The result is $H_c = H_{0c} + H_{Ic}$

$$H_{0c} = m_V\int \mathrm{d}^3p\,\boldsymbol{V}^*(p)\boldsymbol{V}(p) + m_N\int \mathrm{d}^3p\boldsymbol{N}^*(p)\boldsymbol{N}(p) + \int \mathrm{d}^3k\,\omega(k)\boldsymbol{a}^*(k)\boldsymbol{a}(k)\,, \tag{14}$$

$$H_{Ic} = \frac{1}{L}(\sqrt{1+\lambda^2 L} - 1)\boldsymbol{C} + \frac{\Delta}{L^2}(\sqrt{1+\lambda^2 L} - 1)^2\boldsymbol{E}\,, \tag{15}$$

where

$$\boldsymbol{C} = \int \mathrm{d}^3p\int \mathrm{d}^3k\int \mathrm{d}^3k'\,\mathcal{L}_c^2(\omega,\omega')\boldsymbol{a}^*(k')\boldsymbol{N}^*(p-k')\boldsymbol{N}(p-k)\boldsymbol{a}(k)\,, \tag{15a}$$

$$\mathcal{L}_c(\omega,\omega') = F(\omega)\frac{\bar{f}(\omega')}{\sqrt{2(2\pi)^3\omega'}} + \frac{f(\omega)}{\sqrt{2(2\pi)^3\omega}}\bar{F}(\omega')\,, \tag{15b}$$

and $\boldsymbol{E}$ has a similar form with

$$\mathcal{L}_E(\omega,\omega') = F(\omega)\bar{F}(\omega')\,. \tag{15c}$$

The renormalized $\boldsymbol{V}$-particle mass appears in H_{0c}. The integral Δ in H_{Ic} is the same factor as appears in the mass renormalization; however, its presence in H_{Ic} is not associated with an energy shift or cloud effects. None of the

terms in H_{Ic} lead to persistent perturbation effects; in terms of clothed variables there is no longer a cloud of virtual particles.

The S-matrix elements and other physical quantities calculated from H and $H_c = H$ are identical since the clothed and bare operators are related by a unitary transformation which does not change the boundary conditions of eigenvectors. We have verified this fact explicitly for $\boldsymbol{N}$-$\boldsymbol{\theta}$ scattering using only H_c. It is instructive to compare the scattering eigenstates in the two representations. For brevity we discuss $\boldsymbol{N}$-$\boldsymbol{\theta}$ scattering only.

The $\boldsymbol{N}$-$\boldsymbol{\theta}$ incoming and outgoing eigenstates are [9,10,2]

$$(16a)\qquad |N_p, \theta_k\rangle_\pm = |N_p, \theta_k\rangle + \frac{\lambda}{1+\lambda^2 L}\,\frac{F(\omega)}{1+(\lambda^2/(1+\lambda^2 L))(m_N+\omega-m_V)J^{(\pm)}(\omega)}\cdot$$
$$\cdot\,|V_{p+k}\rangle - \frac{\lambda^2}{1+\lambda^2 L}\,\frac{F(\omega)}{1+(\lambda^2/(1+\lambda^2 L))(m_N+\omega-m_V)J^{(\pm)}(\omega)}\cdot$$
$$\cdot\int\frac{\mathrm{d}^3k f(\omega')}{\sqrt{2(2\pi)^3\omega'}(\omega'-\omega\mp i\varepsilon)}\,|N_{(p+k-k')}\theta_{k'}\rangle\,,$$

where

$$(16b)\qquad J^{(\pm)}(\omega) = \int\frac{|F(\omega')|^2}{(\omega'-\omega\mp i\varepsilon)}\,\mathrm{d}^3k\,;\qquad\qquad J^{(1)}(\omega) = P\int\frac{|F(\omega')|^2}{\omega'-\omega}\,\mathrm{d}^3k'\,;$$

Källén's $h(\omega)/\omega$ corresponds to our

$$1+\frac{\lambda^2}{1+\lambda^2 L}\,(m_N+\omega-m_V)J^{(1)}(\omega)\,.$$

Notice that in terms of bare variables the N-θ scattering eigenstates contain a cloud of virtual V-particles even though neither the N nor the θ particle has a self-energy. These eigenstates can be expressed in clothed variables using (10) and (13). The result is

$$(17)\qquad |N_p, \theta_k\rangle_{c\pm} = |\boldsymbol{N}_p, \boldsymbol{\theta}_k\rangle_c + \frac{F(\omega)}{1+(\lambda^2/(1+\lambda^2 L))(m_N+\omega-m_V)J^{(\pm)}(\omega)}\cdot$$
$$\cdot\int\mathrm{d}^3k'\,\bar{F}(\omega')\left\{\frac{1}{L(1+\lambda^2 L)}\left(\sqrt{1+\lambda^2 L}-1\right)-\frac{\lambda^2}{1+\lambda^2 L}\,\frac{m_N+\omega'-m_V}{\omega'-\omega\mp i\varepsilon}\right\}|\boldsymbol{N}_{p+k-k'}, \boldsymbol{\theta}_{k'}\rangle_c\,.$$

In terms of clothed operators there are no virtual $\boldsymbol{V}$-particles in this state.

The S-matrix, calculated from the Hamiltonian expressed in bare or clothed operators, is

$$(18)\qquad {}_-\langle N_q, \theta_k|N_{q'}, \theta_{k'}\rangle_+ = {}_{-c}\langle \boldsymbol{N}_q, \boldsymbol{\theta}_k|\boldsymbol{N}_{q'}, \boldsymbol{\theta}_{k'}\rangle_{c+} = \delta(q-q')\delta(k-k') -$$
$$-\,2\pi i\delta(q+k-q'-k')\delta(\omega-\omega')\,\frac{\lambda^2}{1+\lambda^2 L}\,\frac{|f(\omega')|^2}{2(2\pi)^3\omega'(m_N+\omega'-m_V)}\cdot$$
$$\cdot\,\frac{1}{1+(\lambda^2/(1+\lambda^2 L))(m_N+\omega'-m_V)J^{(+)}(\omega')}\,,$$

where $|\rangle_\pm$ are the scattering eigenstates with incoming and outgoing waves. The S-matrix, as is well known, contains only the expression $\lambda^2/(1+\lambda^2 L)$ which is there normalized charge squared. It is interesting to note that in equation (17) only the term which contributes to the S-matrix, i.e. the one with the proper singularity, contains the renormalized charge; the first term in the bracket in equation (17) does not contain λ and L in this combination.

Since in terms of clothed operators there are no persistent effects in the theory, the states $|\boldsymbol{N}_p, \boldsymbol{\theta}_k\rangle_c$, and in general the states

$$|\rangle_{as\ c} = |\boldsymbol{V}_{p_1} \dots \boldsymbol{V}_{p_m}; \boldsymbol{N}_{q_1} \dots \boldsymbol{N}_{q_n}; \boldsymbol{\theta}_{k_1} \dots \boldsymbol{\theta}_{k_l}\rangle_c$$

produced by a product of clothed creation operators acting on the physical vacuum, are asymptotically stationary in Van Hove's sense (²), that is

$$\lim_{t_0 \to \pm\infty} {}_c\langle\varphi(t+t_0)|\exp[-itH_c]|\varphi(t_0)\rangle = \int |c(\alpha)|^2 \,\mathrm{d}\alpha\,, \tag{19}$$

where

$$\varphi(t_0) = \int |\alpha\rangle_c \exp[-it_0\varepsilon(\alpha)]c(\alpha)\,\mathrm{d}\alpha\,,$$

is any superposition of clothed states. In terms of clothed operators, these asymptotically stationary states do not contain a virtual cloud of particles. Since the clothed operators are unitarily (or more generally canonically) related to the bare operators and thus satisfy the same commutation rules, these states $|\rangle_{c\ as}$ are an orthonormal (more generally orthogonal) set of vectors.

VAN HOVE defines a set of asymptotically stationary states which differ in a « minimal » way from the bare states (²) in the sense that the only states, Y_α, admixed to a given bare state, $|\alpha\rangle$, are the states which occur in intermediate states in calculating the diagonal parts of operators acting on the given bare state. VAN HOVE argues that the set Y_α is the smallest set of states which, admixed to $|\alpha\rangle$, will allow the construction of an asymptotically stationary state, This minimal property of Van Hove's states is attractive. However, as VAN HOVE points out, it is based upon the original separation of H into H_0 and H_I: it is not invariant under canonical transformation.

In general some of Van Hove's asymptotic states contain a cloud of virtual particles expressed either in terms of bare or clothed operators. For example, Van Hove's asymptotic V-state is the same as the physical V-state and in bare operators has a virtual N-θ cloud (see Eq. (4)). His asymptotic N-θ state is $|N_p, \theta_k\rangle$ which in clothed operators has a cloud of $\boldsymbol{V}$ and $\boldsymbol{N}$-$\boldsymbol{\theta}$ particles,

$$|N_p, \theta_k\rangle = |\boldsymbol{N}_p, \boldsymbol{\theta}_k\rangle_c - \frac{\lambda}{\sqrt{1+\lambda^2 L}} F(\omega)\,|\boldsymbol{V}_{p+k}\rangle_c + \\ + \frac{1}{L}\left(\frac{1}{\sqrt{1+\lambda^2 L}} - 1\right) F(\omega)\int \mathrm{d}^3k'\,\overline{F}(\omega')\,|\boldsymbol{N}_{p+k-k'}, \boldsymbol{\theta}_{k'}\rangle_c\,. \tag{20}$$

Further these states $|\alpha\rangle_{as}$ are not orthogonal; the creation operators which, acting on the physical vacuum, create the states $|\rangle_{as}$ ([11]) do not, together with their adjoints, satisfy the canonical commutation rules. However, VAN HOVE does show that the states $|\alpha\rangle_{as}$ are asymptotically orthonormal ([2]) in the sense that

$$\lim_{t_0\to\pm\infty} \langle\varphi_{as}(t_0)|\varphi'_{as}(t_0)\rangle = \int \bar{c}(\alpha)c'(\alpha)\,\mathrm{d}\alpha\,, \tag{21}$$

where φ_{as}, φ'_{as} are constructed from states $|\alpha\rangle_{as}$ as in the manner indicated below equation (18). Because the states $|\alpha\rangle_{as}$ have the minimal property, the lack of orthogonality of these states is essential for the presence of observable interaction effects (*). In the clothed operator formalism, the states $|\alpha\rangle_{cas}$ which do not have the minimal property, are orthonormal.

The conservation laws in the Lee model, namely, that the sum of the number of V and of N particles is constant, and the difference of the number of N and θ particles is constant, imply that the only intermediate states occurring in N-θ scattering are N-θ states (+). Thus, in discussing N-θ scattering we have to consider only the N and θ terms in H_{0c} and the (quadrilinear) N-θ interaction term in H_{Ic} when investigating new clothing transformations which differ from the original one by an expression in N and θ operators. We have obtained such a family of clothing transformations, each of which can be carried out exactly for the terms relevant to N-θ scattering. For each of these the qualitative properties of the Hamiltonian discussed in Sect. **1** remain, although the specific terms in the Hamiltonian change.

The new clothing transformations are of the form $\exp[iS']\cdot\exp[iS]$, where S is given by equation (8) and S' is given by

$$S' = l(\lambda)\boldsymbol{E}\,, \tag{22}$$

where $l(\lambda)$ is arbitrary. The relevant terms of the Hamiltonian when expressed in terms of the new set of clothed operators,

$$V' = \exp[iS']\cdot\exp[iS]\cdot V\cdot\exp[-iS]\cdot\exp[-iS']$$

(where we have suppressed the dependence of V' on $l(\lambda)$) are $H'_c = H'_{0c} + H'_{Ic}$,

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([11]) N. M. HUGENHOLTZ: *Physica*, **23**, 481 (1957), especially pp. 514-521. HUGENHOLTZ has further elucidated the significance of these states using diagramatic arguments.

(*) We thank Prof. VAN HOVE for calling this to our attention.

(+) That this situation does not hold in general can be seen, for example, by considering V-θ scattering.

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where H'_{0c} has the free field form, and

$$H'_{Ic} = \frac{1}{L}\sqrt{1+\lambda^2 L}\sin\left(l(\lambda)L\right)\cdot \boldsymbol{K} + \tag{23}$$

$$+\frac{1}{L}\left(\sqrt{1+\lambda^2 L}\cos\left[l(\lambda)L\right]-1\right)\boldsymbol{C} - \frac{\Delta}{L^2}\left(2\sqrt{1+\lambda^2 L}\cos\left[l(\lambda)L\right]-2-\lambda^2 L\right)\boldsymbol{E}\,,$$

where $\boldsymbol{K}$ is given by an expression similar to (15a), but with

$$\mathcal{L}_K(\omega,\omega') = i\left(F(\omega)\frac{f(\omega')}{\sqrt{2(2\pi)^3\omega'}} - \frac{f(\omega)}{\sqrt{2(2\pi)^3\omega}}F(\omega')\right). \tag{23a}$$

All of these Hamiltonians lead to the same S-matrix for $\boldsymbol{N}$-$\boldsymbol{\theta}$ scattering as has been verified directly. Thus the different interaction terms are an example of different potentials which lead to the same scattering. In this model there is no reason based on physics to prefer one form of H_{Ic} over any other. For computational purposes, some forms of H_{Ic} may be more convenient, for example, the choice $\cos\left(l(\lambda)L\right) = (1+\lambda^2 L)^{-\frac{1}{2}}$ leads to a form with no term in C:

$$H'_{Ic} = \frac{\lambda}{\sqrt{L}}\boldsymbol{K} + \lambda^2\frac{\Delta}{L}\boldsymbol{E}\,. \tag{24}$$

We conclude this section on the Lee model with a brief discussion of non-unitary clothing transformations. There are both advantages and disadvantages in using non-unitary rather than unitary clothing transformations. The main advantages are that, in some theories such as the Lee model, and the Van Hove-Ruijgrok model (see Sect. 4), the non-unitary transformation changes fewer of the states of the theory while still clothing the bare states (in fact in these cases the non-unitary transformation has the minimal property of Van Hove), and is computationally very much simpler. Some disadvantages are that, although non-unitary transformations accomplish mass renormalization, they do not take care of wave function renormalization, the states obtained not being correctly normalized. The asymptotic states to which one is led are asymptotically stationary and asymptotically orthogonal, but are not orthogonal at finite times. Finally Hermiticity properties of field operators are obscured by these non-unitary transformations.

For the Lee model a simple non-unitary clothing transformation is given by $Y = \exp\,[T]$

$$T = -\lambda\iint F(\omega)a^*(k)N^*(p-k)V(p)\,\mathrm{d}^3p\,\mathrm{d}^3k\,. \tag{25}$$

This transformation does not change the vacuum, θ or N states. The V state becomes

$$(26)\qquad e^{T}|V_p\rangle = |V_p\rangle - \lambda\int \mathrm{d}^3k F(\omega)\,|N_{p-k},\theta_k\rangle = \sqrt{1+\lambda^2 L}\,|V_p\rangle\,.$$

In discussing the clothed operators, we distinguish the transform of the adjoint of V (or N or θ) written $\boldsymbol{V}^{\dagger} = \exp[T]V^*\cdot\exp[-T]$, from the adjoint of the transform of V, written $\boldsymbol{V}^* = (\boldsymbol{V})^* = \exp[-T^*]\cdot V^*\exp[T^*]$, where $\boldsymbol{V} = \exp[T]\cdot V\cdot\exp[-T]$. We denote the adjoint of $\boldsymbol{V}^{\dagger}$ by $\boldsymbol{V}^{\dagger *}$. The symbol $\boldsymbol{V}^{*\dagger}$ is neither used nor defined.

The transformed operators $\boldsymbol{V}^{\dagger}$, $\boldsymbol{V}$ and the corresponding pairs for the N and θ fields satisfy the canonical commutation rules. The same is true for the pairs $\boldsymbol{V}^{\dagger *}$, $\boldsymbol{V}^*$, etc. However, for example, the pair $\boldsymbol{N}$, $\boldsymbol{N}^*$ does not satisfy this anticommutation rule.

All of the transformed operators can be formed exactly by expanding the exponential in multiple commutators. We collect some results of interest:

$$(27)\qquad \boldsymbol{V}^{\dagger}(p) = e^{T}V^*(p)e^{-T} = V^*(p) - \lambda\int \mathrm{d}^3k F(\omega)a^*(k)N^*(p-k)\,,$$

$$(28)\qquad \boldsymbol{N}^{\dagger}(p) = e^{T}N^*(p)e^{-T} = N^*(p)\,,$$

$$(29)\qquad \boldsymbol{a}^{\dagger}(k) = e^{T}a^*(k)e^{-T} = a^*(k)\,,$$

$$(30)\qquad \boldsymbol{V}(p) = e^{T}V(p)e^{-T} = V(p)\,,$$

$$(31)\qquad \boldsymbol{N}(p) = e^{T}N(p)e^{-T} = N(p) + \lambda\int \mathrm{d}^3k F(\omega)a^*(k)V(p+k)\,,$$

$$(32)\qquad \boldsymbol{a}(k) = e^{T}a(k)e^{-T} = a(k) + \lambda\int \mathrm{d}^3p N^*(p-k)V(p)F(\omega)\,.$$

The entire Hamiltonian expressed in these variables can be found in closed form; however, we will write down only the bilinear and trilinear terms, as well as the term responsible for $\boldsymbol{N}$-$\boldsymbol{\theta}$ scattering. The result is $H = H''_c = H''_{0c} + H''_{Ic}$, where H''_{0c} has the free field form, and

$$(33)\qquad H''_{Ic} = \lambda\int \mathrm{d}^3p\int \mathrm{d}^3k\,\frac{f(\omega)}{\sqrt{2(2\pi)^3\omega}}\,\boldsymbol{V}^{\dagger}(p)\boldsymbol{N}(p-k)\boldsymbol{a}(k) + \\ + \lambda^2\int \mathrm{d}^3k\int \mathrm{d}^3k'\int \mathrm{d}^3p\,\frac{f(\omega)}{\sqrt{2(2\pi)^3\omega}}\,\bar{F}(\omega')\boldsymbol{a}^{\dagger}(k')\boldsymbol{N}^{\dagger}(p-k')\boldsymbol{N}(p-k)\boldsymbol{a}(k) + \ldots$$

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Notice that unlike the case of unitary clothing transformations, there is a trilinear term in H''_{Ic}. Further, although the total Hamiltonian is still Hermitian, it is not « manifestly Hermitian », and in fact the individual terms H''_{0c}, H''_{Ic} are *not* Hermitian as can be seen from the fact that the eigenvectors of H_{0c} are not orthogonal although they belong to real eigenvalues (*).

The S-matrix calculated in these variables is the same as before provided one uses the operators $\boldsymbol{N}^*$, $\boldsymbol{a}^*$ to create particles on the left side of the inner product, while the operators $\boldsymbol{N}^\dagger$, $\boldsymbol{a}^\dagger$ are used on the right. One has

$$(34)\qquad (\boldsymbol{N}^*\boldsymbol{a}^*\Psi_0,\, \boldsymbol{N}^\dagger\boldsymbol{a}^\dagger\Psi_0) = (e^{-T^*}N^*a^*e^{T^*}\Psi_0,\, e^{T}N^*a^*e^{-T}\Psi_0) = \\ = (N^*a^*\Psi_0,\, N^*a^*\Psi_0)\,,$$

so that with the rule given above the scalar product, and thus the S-matrix, is invariant under this non-unitary transformation. Alternatively, the S-matrix can be found from the standing wave eigenvector of the Hamiltonian.

4. – The Ruijgrok-Van Hove model.

One of the characteristic features of the Lee model which was analyzed in the previous section is the impossibility of successive emission or absorption of several bosons by the heavy particles. To remedy this situation RUIJGROK and VAN HOVE ([12]) have proposed interesting models for which one can still determine the physical one particle states and the renormalization constants in closed form, and in which it is possible for the nucleons to emit or absorb an arbitrary number of bosons. These models also can give rise (for finite sources) to meson-nucleon scattering. In this section we consider the simplest such theory for which the Hamiltonian is

$$(1)\qquad H = H_0 + H_I\,,$$

$$(1a)\qquad H_0 = m_{0V}\int \mathrm{d}^3p\, V^*(p)V(p) + m_{0N}\int \mathrm{d}^3p\, N^*(p)N^*(p) + \int \mathrm{d}^3k\,\omega(k)a^*(k)a(k)\,,$$

$$(1b)\qquad H_I = \lambda_1\int \mathrm{d}^3p\int \mathrm{d}^3k\, \frac{f(\omega)}{\sqrt{2(2\pi)^3\omega}}\,[V^*(p)N(p-k)a(k) + \text{cc.}] + \\ + \lambda_2\int \mathrm{d}^3p\int \mathrm{d}^3k\, \frac{f(\omega)}{\sqrt{2(2\pi)^3\omega}}\,[N^*(p)V(p-k)a(k) + \text{c.c.}]\,,$$

(*) A simple example of this situation is the 2×2 matrix $\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$ which has eigenvectors $u_\pm = \begin{pmatrix}1\\ \pm 1/\sqrt{2}\end{pmatrix}$ belonging to the real eigenvalues $1 \pm \sqrt{2}$. However the inner product $(u_\mp, u_\pm) \neq 0$.

([12]) M. W. RUIJGROK and L. VAN HOVE: *Physica*, **22**, 880 (1956).

where the operators N, V, a have the same meaning as in the Lee model. Clearly when $\lambda_2 = 0$ one reverts to the Lee model, whereas when $\lambda_1 = \lambda_2$ the scalar field situation obtains. Since in the sequel we shall primarily be concerned with the one particle states, we drop the momentum dependence of the N and V operators.

It is clear from a diagramatic analysis of the interaction that the one V-particle state has the following structure:

$$|V\rangle_c = \sqrt{Z_V}\left\{V^*|0\rangle + \sum_{n=1}^{\infty}\frac{1}{\sqrt{2n!}}\int d^3k_1 \dots \int d^3k_{2n}\cdot \right.$$
$$\cdot \pi_V^{(2n)}(k_1, k_2 \dots k_{2n}) a^*(k_1) a^*(k_2) \dots a^*(k_{2n}) V^*|0\rangle +$$
$$+ \sum_{n=0}^{\infty}\frac{1}{\sqrt{(2n+1)!}}\int d^3k_1 \int d^3k_2 \dots \int d^3k_{2n+1}\cdot$$
$$\left.\cdot \nu_V^{(2+n1)}(k_1, k_2, \dots k_{2n+1}) a^*(k_1) a^*(k_2) \dots a^*(k_{2n-1}) N^*|0\right\}. \tag{2}$$

The amplitudes

$$\pi_V^{(2n)}(k_1, \dots k_{-n}) = \frac{(\lambda_1\lambda_2)^n}{\sqrt{2n!}}\prod_{i=1}^{2n}\frac{f(\omega_i)}{\sqrt{2(2\pi)^3\omega^3(k_i)}}, \tag{3}$$

$$\nu_V^{(2n+1)}(k_1, \dots k_{2n+1}) = -\lambda_1\frac{(\lambda_1\lambda_2)^n}{\sqrt{(2n+1)!}}\prod_{i=1}^{2n+1}\frac{f(\omega_i)}{\sqrt{2(2\pi)^3\omega^3(k_i)}}, \tag{4}$$

satisfy the Schrödinger equation

$$H|V\rangle_c = (m_{0V} + \delta m_V)|V\rangle_c = m|V\rangle_c, \tag{5}$$

where

$$\delta m = -\lambda_1^2 \Delta, \tag{6a}$$

$$\Delta = +\frac{1}{(2\pi)^3}\int\frac{d^3k\,|f(\omega)|^2}{2\omega^2(k)}, \tag{6b}$$

and where the unrenormalized bare mass of the N and V particles have been so adjusted that

$$m_{0N} - m_{0V} = (\lambda_2^2 - \lambda_1^2)\Delta \tag{7}$$

so that the renormalized mass of the V and N particle are the same and equal to m. The physical one N particle state can now be obtained by noting the

symmetry of the theory under the interchange $N \to V$, $\lambda_1 \to \lambda_2$. The explicit forms, Eq. (2)–(4), for the amplitudes allow us to rewrite the physical N and V states as follows

$$(8)\qquad |V\rangle_c = \sqrt{Z_V}\left\{V^* \cosh\left(\sqrt{\lambda_1\lambda_2}\int d^3k F(\omega) a^*(k)\right) - N^*\sqrt{\frac{\lambda_1}{\lambda_2}} \sinh\left(\sqrt{\lambda_1\lambda_2}\int d^3k F(\omega) a^*(k)\right)\right\}|0\rangle\,,$$

$$(9)\qquad |N\rangle_c = \sqrt{Z_N}\left\{N^* \cosh\left(\sqrt{\lambda_1\lambda_2}\int d^3k F(\omega) a^*(k)\right) - V^*\sqrt{\frac{\lambda_2}{\lambda_1}} \sinh\left(\sqrt{\lambda_1\lambda_2}\int d^3k F(\omega) a^*(k)\right)\right\}|0\rangle\,,$$

where

$$(10)\qquad F(\omega) = \frac{f(\omega)}{\sqrt{2(2\pi)^3\omega^3(k)}}\,.$$

The wave function renormalization constants Z_N and Z_V are determined from the requirement that

$$(11)\qquad {}_c\langle V|V\rangle_c = {}_c\langle N|N\rangle_c = 1$$

and are found to be

$$(12)\qquad Z_V = \left(\cosh \lambda_1\lambda_2 L + \frac{\lambda_1}{\lambda_2}\sinh \lambda_1\lambda_2 L\right)^{-1},$$

$$(13)\qquad Z_N = \left(\cosh \lambda_1\lambda_2 L + \frac{\lambda_2}{\lambda_1}\sinh \lambda_1\lambda_2 L\right)^{-1}.$$

L is given by

$$(14)\qquad L = \int d^3k\, |F(\omega)|^2\,.$$

Again due to the absence of pair phenomena

$$(15a)\qquad |k\rangle_c = a^*(k)|0\rangle\,,$$

$$(15b)\qquad H|k\rangle_c = \omega(k)|k\rangle_c\,,$$

so that « bare » and « physical » meson states are identical.

We have not been able to obtain a unitary clothing transformation which generates the physical and one particle states to all orders in λ_1 and λ_2. The following transformation $U = \exp[iS]$ does generate the correct states (including the normalization constants Z_V and Z_N) to fourth order, and has the

correct limits to recover the Lee model and scalar field:

$$S = g_1(\lambda_1, \lambda_2) S_1 + g_2(\lambda_1, \lambda_2) S_2 + g_3(\lambda_1, \lambda_2) S_3 + g_4(\lambda_1, \lambda_2) S_4 \tag{16}$$

$$S_1 = i(V^* N A - N^* V A) , \tag{16a}$$

$$S_2 = i(N^* V A^* - V^* N A) , \tag{16b}$$

$$S_3 = i V^* V (A^{*2} - A^2) , \tag{16c}$$

$$S_4 = i N^* N (A^{*2} - A^2) , \tag{16d}$$

where

$$A = \int \mathrm{d}^3 k \frac{f(\omega)}{\sqrt{2(2\pi)^3 \omega(k)}} a(k) , \tag{17a}$$

$$B = \int \mathrm{d}^3 k F(\omega) a(k) . \tag{17b}$$

To fourth order g_1, g_2, g_3 and g_4 are given by

$$g_1(\lambda_1, \lambda_2) = \lambda_2 \left[1 + \frac{L}{3}(\lambda_1^2 - \lambda_2^2)\right] , \tag{18a}$$

$$g_2(\lambda_1, \lambda_2) = \lambda_1 \left[1 + \frac{L}{3}(\lambda_2^2 - \lambda_1^2)\right] . \tag{18b}$$

$$g_3(\lambda_1, \lambda_2) = \lambda_1 \lambda_2 \frac{L}{12}(\lambda_1^2 - \lambda_2^2) = - g_4(\lambda_1, \lambda_2) , \tag{18c}$$

where due to the above mentioned symmetry property of the theory

$$g_1(\lambda_1, \lambda_2) = g_2(\lambda_2, \lambda_1) ,$$
$$g_3(\lambda_1, \lambda_2) = g_4(\lambda_2, \lambda_1) .$$

It is to be noted that the transformation is no longer trilinear. In fact, we have proved that an S composed only of trilinear terms cannot generate the dressed operators. The Hamiltonian to this order is given by

$$\begin{aligned} H_c = {} & (m_{0V} - \lambda_1^2 \Delta) \boldsymbol{V}^* \boldsymbol{V} + (m_{0N} - \lambda_2^2 \Delta) \boldsymbol{N}^* \boldsymbol{N} + \int \omega(k) \boldsymbol{a}^*(k) \boldsymbol{a}(k) \alpha^3 k + \\ & + (\lambda_1^2 - \lambda_2^2) \left[\frac{1}{2} - \frac{L}{8}(\lambda_1^2 + 7\lambda_2^2)\right] \boldsymbol{N}^* \boldsymbol{N} (\boldsymbol{A}^* \boldsymbol{B} + \boldsymbol{B}^* \boldsymbol{A}) - \\ & - (\lambda_1^2 - \lambda_2^2) \left[\frac{1}{2} - \frac{L}{8}(7\lambda_1^2 + \lambda_2^2)\right] \boldsymbol{V}^* \boldsymbol{V} (\boldsymbol{A}^* \boldsymbol{B} + \boldsymbol{B}^* \boldsymbol{A}) + \\ & + \frac{1}{4}(\lambda_1^2 - \lambda_2^2) \Delta (\boldsymbol{N}^* \boldsymbol{N} + \boldsymbol{V}^* \boldsymbol{V}) \boldsymbol{A}^* \boldsymbol{A} + (\lambda_1^2 + \lambda_2^2) \Delta \boldsymbol{V}^* \boldsymbol{N}^* \boldsymbol{N} \boldsymbol{V} + \\ & + \frac{2}{3} \lambda_1 \lambda_2 (\lambda_1^2 - \lambda_2^2) \Delta (\boldsymbol{N}^* \boldsymbol{N} - \boldsymbol{V}^* \boldsymbol{V}) (\boldsymbol{A}^{*2} + \boldsymbol{A}^2) + \dots . \end{aligned} \tag{19}$$

One again notices that the Hamiltonian does not have any terms which give rise to persistent effects in fourth order. It does have terms which will give rise to such effects in higher order, e.g., the last two terms. A consistent extension of these calculations would of course remove such higher order terms. The occurrence of the term Δ in the $\boldsymbol{N}$-$\boldsymbol{\theta}$ and $\boldsymbol{V}$-$\boldsymbol{\theta}$ scattering terms is once again not associated with cloud effects. The Δ factor in the $\boldsymbol{N}$-$\boldsymbol{V}$ interaction term is due to the fact that both particles are located at the origin and corresponds to a Yukawa potential for zero separation of the particles.

It is interesting to note that here again there exists a very simple non-unitary clothing transformation. Consider, for example,

$$Y = e^T\,, \qquad T = -(\lambda_1 N^* V + \lambda_2 V^* N) R\,, \tag{20}$$

where R is an operator involving the meson operator, e.g.

$$R_1 = \int d^3k F(\omega) a^*(k)\,, \tag{21a}$$

or

$$R_2 = \int d^3k F(\omega)\big(a(-k) - a^*(k)\big)\,. \tag{21b}$$

With either of the above forms for T one finds

$$\boldsymbol{V}^\dagger = e^T V^* e^{-T} = V^* \cosh(\sqrt{\lambda_1\lambda_2}\, R) + \sqrt{\frac{\lambda_1}{\lambda_2}}\, N^* \sinh(\sqrt{\lambda_1\lambda_2}\, R)\,, \tag{22}$$

$$\boldsymbol{N}^\dagger = e^T N e^{-T} = N^* \cosh(\sqrt{\lambda_1\lambda_2}\, R) + \sqrt{\frac{\lambda_2}{\lambda_1}}\, V^* \sinh(\sqrt{\lambda_1\lambda_2}\, R)\,, \tag{23}$$

so that apart from the wave function renormalization factors, the operators $\boldsymbol{V}^\dagger$ and $\boldsymbol{N}^\dagger$ create the correct physical one particle states. Furthermore,

$$e^T|0\rangle = |0\rangle_c = |0\rangle \tag{24}$$

and since for either form of R

$$\boldsymbol{a}^\dagger(k)|0\rangle_c = a^*(k)|0\rangle\,, \tag{25}$$

the clothed one meson state is the same as the bare one meson state.

It should be noted that $\boldsymbol{V}^\dagger$ and $\boldsymbol{N}^\dagger$ are not the adjoints of $\boldsymbol{N}, \boldsymbol{V}$ which are

given by

$$\mathbf{N} = e^{T} N e^{-T} = N \cosh(\sqrt{\lambda_1 \lambda_2}\, R) - \sqrt{\frac{\lambda_1}{\lambda_2}}\, V \sinh(\sqrt{\lambda_1 \lambda_2}\, R), \tag{26}$$

$$\mathbf{V} = e^{T} V e^{-T} = V \cosh(\sqrt{\lambda_1 \lambda_2}\, R) - \sqrt{\frac{\lambda_2}{\lambda_1}}\, N \sinh(\sqrt{\lambda_1 \lambda_2}\, R), \tag{27}$$

but since the clothing transformation is a similarity transformation

$$[\mathbf{V}^\dagger, \mathbf{V}]_+ = [\mathbf{N}^\dagger, \mathbf{N}]_+ = 1\,, \tag{28a}$$

$$[\mathbf{a}(k), \mathbf{a}^\dagger(k')] = \delta(k - k')\,. \tag{28b}$$

The Hamiltonian is readily computed for either of the above forms of R and one finds

$$\begin{aligned} H_c = {} & m(\mathbf{N}^\dagger \mathbf{N} + \mathbf{V}^\dagger \mathbf{V}) + \int \mathrm{d}^3k\, \mathbf{a}^\dagger(k)\mathbf{a}(k)\omega(k) + \\ & + [(\lambda_1 - \lambda_2)\mathbf{V}^\dagger \mathbf{N} + (\lambda_2 - \lambda_1)\mathbf{N}^\dagger \mathbf{V}] \int \mathrm{d}^3k \frac{f(\omega)}{\sqrt{2(2\pi)^3\omega(k)}}\, \mathbf{a}(k) - \\ & + (\lambda_2^2 - \lambda_1^2)(\mathbf{V}^\dagger \mathbf{V} - \mathbf{N}^\dagger \mathbf{N}) \frac{1}{2}\left(\frac{1}{1\,!}\, 2R + \frac{1}{3\,!}\,(\lambda_1\lambda_2)(2R)^3 + \ldots\right) \int \mathrm{d}^3k \frac{f(\omega)}{\sqrt{2(2\pi)^3\omega(k)}}\, \mathbf{a}(k) - \\ & - (\lambda_2^2 - \lambda_1^2)(\lambda_1 \mathbf{N}^\dagger \mathbf{V} - \lambda_2 \mathbf{V}^\dagger \mathbf{N}) \frac{1}{2}\left(\frac{1}{2\,!}\,(2R)^2 + \frac{1}{4\,!}\,(\lambda_1\lambda_2)^2(2R)^4 + \ldots\right) \int \mathrm{d}^3k \frac{f(\omega)\mathbf{a}(k)}{\sqrt{2(2\pi)^3\omega(k)}} + \\ & + \text{terms in } \mathbf{V}^\dagger \mathbf{N}^\dagger \mathbf{N} \mathbf{V}. \end{aligned} \tag{29}$$

The above expression is valid for R of the form (21*a*). If R is taken to be expression (21*b*) then the underlined terms in the second line should be omitted. For the derivation of this Hamiltonian we have again assumed that the bare mesons of N and V particles are related according to (7). The meson terms in the third and fourth line can be summed to sinh and cosh terms. It is clear from the structure of the interaction terms that the above transformation has removed self energy effects to all orders. In particular for R of the form (21*a*) the Hamiltonian is normal ordered in terms of the dressed operators. It, however, has not removed those effects arising from vertex modifications. Thus the term $(\lambda_2^2 - \lambda_1^2)\mathbf{V}^\dagger \mathbf{V} \mathbf{A}^\dagger \mathbf{A}$ gives rise to a direct $\mathbf{V}$-$\boldsymbol{\theta}$ scattering. However, iteration of H_{Ic} will likewise give rise to $\mathbf{V}$-$\boldsymbol{\theta}$ scattering terms. For example, the combination of terms $\lambda_1(\lambda_2^2 - \lambda_1^2)\mathbf{N}^\dagger \mathbf{V} \mathbf{A}^{\dagger 2} \mathbf{A}$ and $\lambda_1 \mathbf{V}^\dagger \mathbf{N} \mathbf{A}$ will give rise to $\mathbf{V}$-$\boldsymbol{\theta}$ scattering in higher orders. In fact, in the limit of point sources, such an iterate will contribute a divergent multiple of the term to the scat-

tering. This contribution is the familiar Z_1 or vertex divergence. The characteristic feature of this type of divergence is that it is associated with real two particle processes. In the present context this possibility remains due to the presence of the trilinear terms $(\lambda_1 V^\dagger N + \lambda_2 N^\dagger V)A$. We could try to eliminate these effects by performing a canonical transformation on the Hamiltonian to remove these terms. We shall, however, not do so, deferring an analysis of the Z_1 divergences to the relativistic case.

5. – Conclusion.

In the preceding sections we have tried to reformulate three simple field theoretic models so that only finite quantities referring to physical (clothed) particles enter in the Hamiltonian description of the theory. Our reformulation of the Lee and Ruijgrok-Van Hove models is incomplete in that the charge renormalization has not been effected. Thus for example, (3.29) for the Hamiltonian in clothed operators for the Ruijgrok-Van Hove model will still give rise, for infinite cut-off, to vertex divergences. However, in the scalar model the fact that $Z_1 = Z_2$ implies that the coupling constant occurring in H_c is the renormalized charge as defined by DYSON (13).

The peculiar features of the charge renormalization in both the Lee model (14) and the Ruijgrok-Van Hove (15) model suggest that in the point source limit of these models charge renormalization has properties different from those in the more interesting relativistic theories. We hope to return to the question of charge renormalization in more realistic theories than the ones treated here in a future publication. Quantum electrodynamics by virtue of Ward's theorem that $Z_1 = Z_2$, as a consequence of gauge invariance, would seem the simplest theory to treat in the spirit of this paper.

Another question which should be investigated in connection with our formulation is the relation of the $U(0, -\infty)$ transformation (16) to clothing transformations, as well as the relation of in, out and other mass shell parts of the renormalized Heisenberg operators to our clothed Heisenberg operators.

Finally, we note several features which may be expected to hold in the relativistic case when the theory is expressed in terms of clothed operators. One of these is the fact that the canonical form of the twofold vacuum expec-

(13) F. J. DYSON: *Phys. Rev.*, **75**, 1736 (1949).

(14) G. KÄLLÉN: CERN/T/GK3/.

(15) See in this connection G. DELL'ANTONIO and F. DUIMIO: *Nuovo Cimento*, **6**, 751 (1957).

(16) M. GELL-MANN and F. LOW: *Phys. Rev.*, **84**, 350 (1951). We should like to acknowledge an interesting conversation regarding this point with F. LOW.

tation value of the dressed operators will take the form

(1a) $$_c\langle 0 | T(\psi(x_\mu)\psi(x'_\mu)) | 0\rangle_c = iS_F(x_\mu - x'_\mu;\, m)\,,$$

(1b) $$_c\langle 0 | T(\varphi(x_\nu)\,\varphi(x'_\nu)) | 0\rangle_c = i\Delta_F(x_\nu - x'_\nu;\, \mu^2)\,,$$

where m and μ are the renormalized masses of the Fermion and Boson particles respectively. The right hand side of these expressions are the free field Green's functions, Eq. (1) now expressing the fact that ψ and φ create physical particles which propagate in a manner characteristic for a free particle of that mass and spin.

The Hamiltonian for the relativistic case will again have the properties enunciated in Sect. **1**. Thus, for example, quantum electrodynamics when expressed in terms of clothed operators will have a Hamiltonian which no longer contains any trilinear terms such as $\psi(x)\gamma^\mu\psi(x)A_\mu(x)$, but now contains an infinite series of normal ordered terms corresponding to real processes. A new feature of the relativistic theory, however, is that if we require the theory to be microscopically causal, then very intimate connections will exist between the c-number coefficients of the various normal ordered terms which make up the Hamiltonian. These connections will be the counterpart of dispersion relations in our context.

In conclusion, we note that our starting point was a Hamiltonian with the conventional trilinear interaction terms, originally suggested by correspondence principle and classical arguments originating in electrodynamics. We then proceeded to reformulate the theory in such a way that persistent effects no longer appear. It would be interesting to invert the process and inquire whether it is possible to formulate a relativistically covariant, causal, divergence free Hamiltonian theory involving only clothed operators.

* * *

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Note added in proof.

Dr. Drell has informed us that Dr. Zachariasen and he have independently arrived at the idea of formulating field theories in terms of clothed operators and have given a formulation essentially equivalent to Sect. **1**.

RIASSUNTO (*)

Si esamina l'idea di introdurre operatori fisici nella teoria quantistica dei campi, cioè operatori che creano e distruggono gli autostati di una particella dell'hamiltoniana totale. Nelle teorie di taglio questi operatori fisici sono affini agli operatori della teoria non rinormalizzata di Schrödinger in una trasformazione di similitudine. In termini. di tali operatori fisici la nube virtuale di particelle circondante una particella fisica quale la si concepisce nelle attuali discussioni sulla teoria dei campi non appare più corrispondente, nella terminologia di Van Hove, all'eliminazione di effetti persistenti. La rinormalizzazione delle funzioni di massa e d'onda si ottiene automaticamente per mezzo di trasformazioni di rivestimento unitarie e il consueto programma di rinormalizzazione si estende alla rinormalizzazione degli stati comparenti nella teoria. Si discute la forma dell'hamiltoniana espressa in termini di operatori fisici e si applicano tali idee al modello scalare statico, al modello di Lee ed a quello di Ruijgrok-Van Hove.

(*) *Traduzione a cura della Redazione.*

Non-local Theory of Elementary Particles.

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Summary. — It is noted that if the elementary particles are described by non-local fields then the principle of conservation of parity is regained. A modified form of a previous model of the internal co-ordinate wave equation is shown to give satisfactory agreement with the observed mass spectrum, decays and cross-sections of the seven known unstable particles and it predicts the occurrence of more heavy unstable particles. Then a simple one dimensional model is constructed and it gives the same mass spectrum as before and the original Pais selection rules.

1. – Introduction.

To account for the large associated production of the unstable elementary particles as well as their comparatively slow decay Pais (1) suggested that the strong interactions, which allow the former reactions, are forbidden for the latter processes, and postulated odd even selection rules for the interacting fields. By enlarging the isotopic space by means of a strangeness quantum number Gell-Mann and Pais (2) and Nishijima (3) have obtained a satisfactory set of rules which give agreement with all the observed results and have used them to make some successful predictions. Then d'Espagnat and Prentki (4) and Schwinger (5) have given physical interpretation of the new

(1) A. Pais: *Phys. Rev.*, **86**, 663 (1952).

(2) M. Gell-Mann and A. Pais: *Proceedings of the* 1954 *Glasgow Conference on Nuclear and Meson Physics* (New York, 1955).

(3) K. Nishijima: *Progr. Theor. Phys.*, **13**, 285 (1955).

(4) B. d'Espagnat and J. Prentki: *Nucl. Phys.*, **1**, 33 (1956).

(5) J. Schwinger: *Phys. Rev.*, **104**, 1164 (1956).

quantum number and other possible assignments of it have been given by SACHS (6), TIMONO (7) and PAIS (8), who has also suggested a wave equation which defines such quantum numbers and has obtained a qualitative mass spectrum from it.

On the other hand YUKAWA (9) proposed to augment the quantum numbers of a elementary particle by giving it internal degrees of freedom which can also be utilized to obtain convergent field theories. A set of internal co-ordinate wave equations from which it was attempted to obtain a quantitative mass spectrum of the elementary particles is given in SEN (10) and when these wave equations are a little modified the matrix elements for the decay of the unstable elementary particles by means of strong interactions are found to vanish and their comparative stability as well as the large associated production are obtained. Such a demonstration appears the more convincing as all the intrinsic properties of the elementary particles are established from one set of wave equations and, furthermore, an advantage of the non-local theory is that it makes possible the reinstatement of the principle of conservation of parity (see LEE and YANG (11), SALAM (12), LANDAU (13) and YANG (14)) without further assumptions.

The wave equations of I are modified by using, instead of the hard sphere boundary conditions, the RAYLEIGH (15,16) boundary conditions where the wave functions are represented by the terms of the Schlömilch series. This facilitates the evacuation of the interaction matrix elements and they are found not to vanish only when internal quantum numbers of the wave functions of the interacting fields can form the sides of a triangle that is that the magnitude of any one of them lies between the absolute values of the difference and the sum of the other two quantum numbers. Our procedure throughout I has been to construct a consistent model of the internal co-ordinate wave equations which gives correct results for a set of observations and the choice of these boundary conditions is an extension of the above procedure

(6) R. G. SACHS: *Phys. Rev.*, **99**, 1573 (1955).

(7) J. TIMONO: *Nuovo Cimento*, **6**, 69 (1957).

(8) A. PAIS: *Physica*, **19**, 869 (1953).

(9) H. YUKAWA: *Phys. Rev.*, **77**, 219 (1950); **80**, 1047 (1950); **91**, 415, 416 (1953); *Rev. Mod. Phys.*, **29**, 213 (1957).

(10) P. SEN: *Nuovo Cimento*, **3**, 612 (1956). This paper is called I.

(11) T. D. LEE and C. N. YANG: *Phys. Rev.*, **102**, 290 (1956); **104**, 254 (1956); **105**, 1671 (1957).

(12) A. SALAM: *Nuovo Cimento*, **5**, 299 (1957).

(13) L. LANDAU: *Nucl. Phys.*, **3**, 127 (1957).

(14) C. N. YANG: *Rev. Mod. Phys.*, **29**, 231 (1957).

(15) LORD RAYLEIGH: *Phys. Mag.*, (6) **21**, 567 (1911).

(16) G. N. WATSON: *Theory of Bessel Functions* (Cambridge, 1944).

to include the slow decay and the large associated production of the elementary particles within the model from which their mass spectrum is obtained. The significance of the approximations made can be found by constructing several satisfactory models and then determining their common properties.

We shall also construct a simple one dimensional model of the internal co-ordinate wave equations which is of interest as it gives the PAIS [1] selection rules and also makes the procedure more obvious. This opportunity is also utilized to modify the mass term of the wave equations in order to obtain better agreement with the observed masses which have been tabulated by CROWE [17] and to include the recent discovery of Σ^0 particle by PLANO, SAMIOS, SCHWARTZ and STEINBERGER [18]. Further agreement between the observed and the calculated mass spectra is obtained by noting that some calculated unstable particles, which have not been observed, allow rapid decay through strong interactions.

2. – Internal co-ordinate wave equations.

In order to obtain suitable selection rules and to improve the mass spectrum obtained in I by including the recently discovered Σ^0 particle and by eliminating the particle of mass and spin 0 whose existence affects the decay schemes let the internal co-ordinate wave equations be modified so that they become

$$(1)\quad \left|\begin{aligned}
&\left\{\square(r_\mu) + \varkappa^2 - \lambda''\varkappa_0\varkappa\right\}\pi_\varkappa(r)\,T^0_0(\varphi,\vartheta_1,\vartheta_2) = 0\,,\\
&\left\{\square(r_\mu) - (5/4)r^{-2} + \varkappa^2 - \lambda'\varkappa_0\varkappa + a^{-2}\right\}\psi_\varkappa^{e^-,+}(r)\,T^\varrho_{\frac12}(\varphi,\vartheta_1,\vartheta_2) = 0\,,\\
&\left\{\square(r_\mu) - (5/4)r^{-2} + \alpha'^2\left((\varkappa-\varkappa_0)^2 + 2\lambda'\varkappa_0(\varkappa-\varkappa_0)\right) + a^{-2}\right\}\psi_\varkappa^0(r)\,T^\varrho_{\frac12}(\varphi,\vartheta_1,\vartheta_2) = 0\\
&\left\{\square(r_\mu) - (5/4)r^{-2} + \alpha'^2(\varkappa-\varkappa_0)^2 + a^{-2}\right\}\psi_\varkappa^{+,-}(r)\,T^\varrho_{\frac12}(\varphi,\vartheta_1,\vartheta_2) = 0\,.
\end{aligned}\right.$$

Here the light and the heavy particles of spin $\frac12$ are denoted by $\psi_\varkappa^{e^-,+}(r_\mu)$ and $\psi^{+,0,-}(r_\mu)$ respectively, the mass as well as the charge factors of I (33) have been modified, a scale factor α' has been introduced in order to eliminate the π-particle of mass zero without disturbing the mass spectrum of the heavy particles, and additional terms with factors a^{-2} and $(5/4)r^{-2}$ have been added to prevent the degeneration of the stable particle wave functions into null functions and to obtain a favourable set of selection rules for the interaction matrix elements. The wave equations of particles of spin 1 and of light neutral

[17] K. M. CROWE: *Nuovo Cimento*, **5**, 541 (1957).

[18] R. PLANO, N. SAMIOS, M. SCHWARTZ and J. STEINBERGER: *Nuovo Cimento*, **5**, 216 (1957).

particles of spin $\frac{1}{2}$ have not been written as all the known unstable particles can be assigned spins 0 or $\frac{1}{2}$ and light neutral unstable particles of spin $\frac{1}{2}$ have also not been found. The last three equations of (1) can be combined in the form

$$\begin{aligned}(2)\qquad &\{\Box(r_\mu) - (5/4)r^{-2} + \big(1 + (\alpha'^2 - 1)\,\delta_{z(z-1),0}\big)\big((\varkappa - \varkappa_0\,\delta_{z(z-1),0})^2 + \\ &+ 2\lambda'\varkappa_0(1-z)\big(1 + (5/4)z\big)(\varkappa - \varkappa_0\,\delta_{z(z-1),0}) + a^{-2}\}\psi_\varkappa(r)\,T^{\varrho}_{\frac{1}{2}}(\varphi, \vartheta_1, \vartheta_2) = 0\,,\end{aligned}$$

provided it is understood that the particle and its antiparticle have the same mass spectrum and opposite charges. Then we obtain

$$(3)\qquad \left\{\begin{aligned} &\pi_\varkappa(r) = N_0(\varkappa)r^{-1}J_1\big((\varkappa^2 - \lambda''\varkappa_0\varkappa)^{\frac{1}{2}}r\big)\,,\\ &\psi^{e(-,+)}_\varkappa(r) = N^-_{\frac{1}{2}}(\varkappa)r^{-1}J_1\big((\varkappa^2 - \lambda'\varkappa_0\varkappa + a^{-2})^{\frac{1}{2}}r\big)\,,\\ &\psi^0_\varkappa(r) = N^0_{\frac{1}{2}}(\varkappa)r^{-1}J_1\big(((\varkappa-\varkappa_0)^2 + 2\lambda'\varkappa_0(\varkappa-\varkappa_0) + (\alpha'\,a)^{-2})^{\frac{1}{2}}\alpha' r\big)\,,\\ &\psi^{+,-}_\varkappa(r) = N^+_{\frac{1}{2}}(\varkappa)r^{-1}J_1\big(((\varkappa-\varkappa_0)^2 + (\alpha'\,a)^{-2})^{\frac{1}{2}}\alpha' r\big)\,.\end{aligned}\right.$$

Now in order to evaluate the transition matrix elements we shall replace the hard sphere boundary conditions, that is the vanishing of the wave functions outside a four dimensional sphere of radius a, by the RAYLEIGH [15,16] boundary conditions so that the wave functions are now represented by the terms of the Schlömilch series. For instance, for particles of spin 0

$$(4)\qquad \pi_{\varkappa_m}(r) \equiv \pi_m(r) = N_0(\varkappa_m)r^{-1}J_1(mr/a)\,,\quad \varkappa_m^2 - \lambda''\varkappa_0\varkappa_m = m^2a^{-2}\,,\quad m = 1, 2, \ldots,$$

and for the neutral heavy particles of spin $\frac{1}{2}$

$$(5)\qquad \left\{\begin{aligned} &\psi^0_{\varkappa_n}(r) \equiv \psi^0_n(r) = N^0_{\frac{1}{2}}(\varkappa_n)r^{-1}J_1(nr/a)\,,\\ &\alpha'^2\big((\varkappa_n - \varkappa_0)^2 + 2\lambda'\varkappa_0(\varkappa_n - \varkappa_0) + (\alpha' a)^{-2}\big) = n^2a^{-2}\,,\qquad n = 1, 2, \ldots\,.\end{aligned}\right.$$

The mass spectrum of the elementary particles may be obtained from the equations (4) and (5) and the corresponding equations for $\psi^{+,-}_\varkappa(r)$ and is shown in the Table I. Here the constants λ', λ'', a^{-2} and α' have been chosen to be 1.127, -3.028, $1.574\cdot 10^6\,m_e^2$ and .0556 so that ψ^e_1, π_1, π_2 and ψ^0_2 particles have the masses 207, 270, 965 and 2182 m_e respectively. Thus in order to, obtain a satisfactory mass spectrum for the seven known unstable elementary particles four arbitrary constants compared to three in I are used here and the mass terms of the wave equations have also been varied by arbitrarily chosen charge factors, which become necessary in order to account for the

TABLE I. – *The calculated masses of elementary particles.*

Wave function \ n	1	2	3	4	5 m_e
$\pi^{+,0,-}$	270	965	1899	3129	—
$\psi^{e(-,+)}$	0.207	2 280 (*)	4 102 (*)	—	—
ψ^0	1 836	2 182	2 491	2 794	—
$\psi^{+,-}$	1 836	2 349	2 673	2 983	—

(*) Denotes the particles for which rapid decay is allowed by the selection rules (8) and (17)

charge asymetry due to the occurrence of ψ_2^0, and the validity of the model remains undecided. But although its predictions are unreliable we shall obtain strong supporting evidence for our scheme and the assignment of the quantum numbers of the Table I in the calculations of the selection rules for the interaction field in the next section.

3. – The transition matrix elements.

The internal co-ordinate factor of the interaction amongst the π and ψ fields has been taken to be of the form

$$\overline{\psi}_{n_1}(r_\mu)\pi_m(r_\mu)\psi_{n_2}(r_\mu)\,, \tag{6}$$

and hence its matrix element contains the factor

$$M(n_1\,|\,m\,|\,n_2)=\int\limits_0^\infty J_1(n_1r/a)J_1(mr/a)J_1(n_2r/a)\,\mathrm{d}r\,. \tag{7}$$

This integral is evaluated in WATSON [16] and it vanishes if

$$\text{either}\quad m\leqslant|n_1-n_2|\qquad\text{or}\qquad m\geqslant|n_1+n_2|\,. \tag{8}$$

Other variants of these selection rules, which however appear to be less satisfactory, can be obtained, for instance, by letting the ψ wave function to be represented by $r^{-1}J_{\frac{3}{2}}(\sqrt{4/3}\,nr/a)$ so that the relation (8) is replaced by

$$\text{either}\quad m\leqslant|n_1-n_2|\qquad\text{or}\qquad m=2\,,\quad n_1=n_2=1\,.$$

Then the decays $\pi_2\to\pi_1+\pi_1$ and $\pi_3\to\psi_1^e+\psi_1^e$ are allowed and other considerations remain unaltered.

The selection rules (8) forbid the interactions

$$\overline{\psi}_1(r\ )\pi_1(r_\mu)\psi_2(r_\mu)\,, \qquad \overline{\psi}_1(r_\mu)\pi_2(r_\mu)\psi_1(r_\mu)\,, \tag{9}$$

which prevent strong interaction decay ψ_2^0, and although the decays $\psi_2^{+,-} \to \psi_2^0 + \pi_1^{+,-}$, $\psi_3^{+,-} \to \psi_3^0 + \pi_1^{+,-}$, $\psi_3 \to \psi_2 + \pi_2$ are allowed by the selection rules they are forbidden by the principle of conservation of energy for sufficiently low initial energies. The observed and not the calculated masses must be used for the determination of the limiting energies. Then the decay $\psi_2 \to \psi_1 + \pi_2$ is allowed and therefore the particle will remain unobserved. The above scheme also predicts the existence of the slow decay particles π_3, ψ_4^0, ...

Gell-Mann and Pais [2] have tabulated the reactions amongst the π and ψ particles and by the use of the selection rules (8) the reactions

$$\left\{\begin{aligned} &\pi_1 + \psi_2 \to \psi_3 + \pi_2\,,\\ &\pi_2 + \psi_2 \to \psi_3 + \pi_1\,,\\ &\psi_2 + \psi_2 \to \psi_2 + \psi_3 + \pi_2\,,\end{aligned}\right. \tag{10}$$

$$\left\{\begin{aligned} &\psi_2 + \psi_2 \to \psi_2 + \psi_2\,,\\ &\psi_2 + \psi_3 \to \psi_2 + \psi_3\,,\end{aligned}\right. \tag{11}$$

$$\left\{\begin{aligned} &\psi_2 + \psi_2 \to \psi_3 + \psi_3\,,\\ &\qquad \pi_2 \to \pi_1 + \pi_1\,,\end{aligned}\right. \tag{12}$$

are seen to be allowed, while the reactions

$$\left\{\begin{aligned} &\pi_1 + \psi_2 \to \pi_1 + \psi_3\,,\\ &\psi_2 + \psi_2 \to \psi_2 + \psi_3\,,\\ &\pi_1 + \psi_2 \to \psi_2 + \pi_2\,,\\ &\psi_2 + \psi_2 \to \psi_2 + \psi_2 + \pi_2,\end{aligned}\right. \tag{13}$$

are forbidden. The only contradiction to the observed cross-sections occurs for the reactions (12) which are observed to be forbidden and to account for this a further selection rule seems to be required such that it involves the sums of the initial and final quantum numbers like the principle of conservation of angular momentum for the external co-ordinates.

Consider the reaction

$$\psi_{n_{i1}} + \psi_{n_{i2}} + \pi_{m_i} \to \psi_{n_{f1}} + \psi_{n_{f2}} + \pi_{m_f} \,. \tag{14}$$

Now we shall postulate that the wave function of the combined initial ψ particles in suitable relative co-ordinates is also a Bessel function of order 1 and the total internal quantum number N_i in its argument describes a suitable mass relation so that

$$\psi_{N_i}(r_\mu) = N^+_{\frac{1}{2}}(\varkappa_{N_i}) r^{-1} J_1(N_i r/a)\,, \qquad N_i = n_{i1} + n_{i2} - 1. \tag{15}$$

Similarly let initial and final π particles combine to be represented by the wave function

$$\pi_M(r_\mu) = N_0(\varkappa_M) r^{-1} J_1(Mr/a)\,, \qquad M = m_i + m_f - 1. \tag{16}$$

Furthermore if no $\pi(\psi)$ particles occur in the initial and the final states of the reaction (14) then, since $M(N_{i,f}) = 0$ is physically insensible, let $M(N_{i,f})$ tend to an arbitrarily small and positive number ε. Then the reaction (14) is forbidden if

$$\text{either} \quad M \leqslant |N_i - N_f| \quad \text{or} \quad M \geqslant |N_i + N_f|\,. \tag{17}$$

Now the selection rules (17) which supplement the selection rules (8) will in general also allow the previously allowed reactions (10) and (11) but will forbid the reactions (12) due to different arguments of the initial and final state Bessel functions, and this completes the demonstration of the assertions made in Sect. **1**.

4. – One dimensional model.

Let us now construct a one dimensional model of the wave equations (1) which is of interest as it leads to strong interaction selection rules originally postulated by Pais (1) and it also serves to clarify the physical picture of the more accurate model given above. Let the equations (1) be replaced by

$$(18) \quad \left\{\begin{aligned}
&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \varkappa^2 - \lambda''\varkappa_0\varkappa\right\}\pi_\varkappa(r) = 0\,,\\
&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \varkappa^2 - \lambda'\varkappa_0\varkappa + a^{-2}\right\}\psi_\varkappa^{e(-,+)}(r) = 0\,,\\
&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \alpha'^2((\varkappa-\varkappa_0)^2 + 2\lambda'\varkappa_0(\varkappa-\varkappa_0)) + a^{-2}\right\}\psi_\varkappa^0(r) = 0\,,\\
&\left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \alpha'^2(\varkappa-\varkappa_0)^2 + a^{-2}\right\}\psi_\varkappa^{+,-}(r) = 0\,,
\end{aligned}\right.$$

so that

$$
(19)\qquad\left\{\begin{aligned}
&\pi_\varkappa(r) &&= N_0(\varkappa)\sin\left((\varkappa^2-\lambda''\varkappa_0\varkappa)^{\frac{1}{2}}r\right),\\
&\psi_\varkappa^{e'(-,+)}(r) &&= N_{\frac{1}{2}}^{-}(\varkappa)\sin\left((\varkappa^2-\lambda'\varkappa_0\varkappa+a^{-2})^{\frac{1}{2}}r\right),\\
&\psi_\varkappa^{0}(r) &&= N_{\frac{1}{2}}^{0}(\varkappa)\sin\left(((\varkappa-\varkappa_0)^2+2\lambda'\varkappa_0(\varkappa-\varkappa_0)+(\alpha' a)^{-2})^{\frac{1}{2}}\alpha' r\right),\\
&\psi_\varkappa^{+,-}(r) &&= N_{\frac{1}{2}}^{+}(\varkappa)\sin\left(((\varkappa-\varkappa_0)^2+(\alpha' a)^{-2})^{\frac{1}{2}}\alpha' r\right),
\end{aligned}\right.
$$

and let the boundary conditions be that the wave functions vanish for $r \geqslant a$. Then we obtain, for instance for $\pi_\varkappa(r)$

$$
(20)\quad \pi_{\varkappa_m}(r) \equiv \pi_m(r) = N_0(\varkappa_m)\sin(m\pi r/a)\,,\quad \varkappa_m^2-\lambda''\varkappa_0\varkappa_m = m^2\pi^2/a^2\,,\quad m=1,2,\ldots,
$$

and hence the same spectrum as before is obtained and is shown in Table I.

To obtain the selection rules for the interacting fields, the integral

$$
(21)\qquad M_1(n_1\,|\,m\,|\,n_2) = \int_0^a \sin(n_1\pi r/a)\sin(m\pi r/a)\sin(n_2\pi r/a)\,\mathrm{d}r\,,
$$

has to be evaluated and it vanishes provided

$$
(22)\qquad n_1+m+n_2 \quad \text{is even}\,,
$$

which is the Pais postulate. These selection rules lead to associated production but allow the decay $\psi_3 \to \psi_1 + \pi_1$.

5. – Conservation of parity in non-local theory.

In non-local theory the paradox of non-conservation of parity which has been investigated by YANG and LEE (11), SALAM (12), LANDAU (13), and YANG (14). is resolved in a particularly simple and physically sensible manner, as only the total wave function, which is the product of the internal and external co-ordinate wave functions, is now required to be covariant with respect to the general Lorentz transformations and it is sufficient that each of the factors of the total wave function be covariant for the proper Lorentz transformations and that the parity of the total wave function be a good quantum number. Thus we obtain results similar to those of LANDAU (13) and YANG (14) who by enlarging the spacetime reflections to include simultaneous charge conju-

gation, have concluded that the combined parity is a good quantum number. Thus, in our model the mutually exclusive π_2 decays are obtained by noting that, for instance, the parity of the internal co-ordinate wave functions can be conserved as well as not conserved provided the external parity which conserves their product is available. These decay schemes can also be further modified by correlating charge and intrinsic parity.

6. – Conclusion.

By describing elementary particles by a non-local field the conservation of parity is regained and all the intrinsic properties of the unstable particles that is their mass spectrum cross-sections and comparative stability, can be derived from one set of internal co-ordinate wave equations. It consists of four wave equations with four arbitrary constants and these are adjusted to give the masses and internal quantum numbers of the stable and the seven known unstable particles. Extra rapidly decaying and hence unobservable particles such as ψ_2 are found to occur in the model as well as the comparatively stable particles π_0, ψ_4^0 and $\psi_4^{+,-}$ provided the mass difference $|\psi_4^{+,-} - \psi_4^0|$ is less than $\varkappa_{\pi_1}$. But due to its large arbitrariness its predictions are not reliable, for instance the sequence of quantum numbers, n^2 in (4) and (5) can be replaced by the sequence $(5n+1)$: 1, 6, ..., and then the decay $\psi_4 \to \psi_3 + \pi_1$ is allowed.

The interactions $\psi_1\pi_1\psi_2$ and $\psi_1\pi_2\psi_1$ are found to be forbidden by the selection rules that the quantum numbers of the allowed matrix elements form the sides of a triangle. Therefore the reactions (10) and (11) are allowed and the reactions (12) and (13) are forbidden. Furthermore the strong interaction decay of ψ_2^0 is forbidden and although the selection rules allow the decays $\psi_2^{+,-} \to \psi_3^0 + \pi_1^{+,-}$, $\psi_3^{+,-} \to \psi_3^0 + \pi_1^{+,-}$, $\psi_3 \to \psi_2 + \pi_2$ they are forbidden by conservation of energy for sufficiently low initial energy. Therefore we conclude that the model gives satisfactory agreement with the observed results.

We note that the isotopic spin is conserved in this model and that as the internal quantum number is not necessary to establish the stability of the unstable particles for the electromagnetic interaction, it has not been considered here, but it can be utilized to obtain better agreement with the observed mass spectrum. Perhaps a more attractive model, which would be closer to YUKAWA's ( ) proposal is to form only two internal coordinate wave equations for the bosons and the fermions, and to consider the total internal quantum numbers to be the sum of internal radial and spin quantum numbers. But since the photons and the π-mesons of spin 0 and mass 270 must both be in the lowest quantum state, degeneracy will occur and for instance there will be mesons of spin 0 and mass 0 and of spin 1 and mass 270.

RIASSUNTO (*)

Si fa notare che descrivendo le particelle elementari per mezzo di campi non locali il principio della conservazione della parità riacquista la sua validità. Si dimostra che una forma modificata di un precedente modello dell'equazione d'onda a coordinate interne dà un accordo soddisfacente collo spettro di massa, i decadimenti e le sezioni d'urto osservati per le sette particelle instabili conosciute e predice l'esistenza di altre particelle instabili pesanti. Si costruisce poi un semplice modello unidimensionale che dà lo stesso spettro di massa di cui sopra e le regole di selezione originali di Pais.

(*) *Traduzione a cura della Redazione.*

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On a Geometrical Theory of the Electromagnetic Field.

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Summary. — Weyl's (1929) geometrical theory of gauge (phase) transformations for fermions and the electromagnetic field is generalized so as to apply to bosons as well. This is done by introducing *complex* base vectors. The resulting theory is closely related to the Einstein-Schrödinger theory in its Hermitian form, and enables one to identify the electromagnetic field tensor. Two physical consequences are deduced from the theory: *a*) The charges of all bosons are integer multiples of a basic charge. *b*) The direct interaction between a particle and the electromagnetic field is invariant under space reflexions and time reflexions. A theoretical criterion is suggested (based on the argument leading to *b*)) for determining which interactions are parity conserving.

1. – Introduction.

The aim of this paper is to propose a new geometrical theory of the electromagnetic field (1).

Since many geometrical theories have already been proposed, some justification is needed for introducing yet another one. The justification lies in the fact that the theory described here is not just a rewriting of the Einstein and Maxwell theories — it leads to new physical consequences. These are that the charges of all bosons must be integer multiples of some basic charge (2), and that the direct interaction between a particle and the electromagnetic

(1) A brief account of the theory has already been given, *Phys. Rev.*, **107**, 632 (1957). However, this reference contains errors, and is superseded by the present paper.

(2) The need for such a result has been emphasized by P. A. M. Dirac: *Phys. Rev.*, **74**, 817 (1948), and E. P. Wigner: *Proc. Amer. Phil. Soc.*, **93**, 521 (1949), for both bosons and fermions.

field must be invariant under space and time reflexions. Both these results are in agreement with observation (³).

The theory consists of a generalization of Weyl's second attempt (⁴) at a geometrical theory of electromagnetism, in which the gauge transformations of his original theory (⁵) became phase transformations (⁶).

Weyl's formalism will be recalled in the next section, but it will be convenient to mention here its main advantages and disadvantages. These remarks also apply to other theories of the same type (⁷).

Advantages.

(i) The electromagnetic field is « generated » by invariance arguments which specify the form of its coupling to matter (except possibly for terms depending on the field strength, which are usually excluded—the so-called principle of minimal electromagnetic coupling). This is in exact analogy with the gravitational field (where the corresponding principle is called the principle of equivalence).

(ii) Despite its geometrical basis, the theory permits different particles to move in different orbits in the same electromagnetic field. This, of course, is in direct contrast with purely gravitational motion, and corresponds to the empirical fact that the charge-mass ratio e/m is not the same for all particles.

Disadvantages.

(i) The electromagnetic potential is part of the *spin* connexion only, and not of the affine connexion of the Riemannian space. As a result, only fermions can interact with the electromagnetic field, whereas we know empirically that some bosons interact with it too. In the conventional treatment of the electromagnetic field (⁸) the boson interaction is introduced by supposing that the boson field φ can undergo a phase transformation without altering the field equations.

This supposition is based on the assumption that φ occurs in the Lagran-

(³) T. D. LEE and C. N. YANG: *Phys. Rev.*, **104**, 254 (1956), discuss the evidence for parity conservation in electromagnetic interactions.

(⁴) H. WEYL: *Proc. Nat. Acad. Sci.*, **15**, 323 (1929); *Zeits. f. Phys.*, **56**, 330 (1929); *Phys. Rev.*, **77**, 699 (1950).

(⁵) H. WEYL: *Space-Time-Matter* (New York, 1951), p. 282.

(⁶) Unfortunately the phase transformations are usually still called gauge transformations.

(⁷) V. FOCK: *Zeits. f. Phys.*, **57**, 261 (1929); E. SCHRÖDINGER: *Berl. Ber.*, 105 (1932); L. INFELD and B. L. v. DER WAERDEN: *Berl. Ber.*, 380 (1933); W. L. BADE and H. JEHLE: *Rev. Mod. Phys.*, **25**, 714 (1953); P. G. BERGMANN: *Phys. Rev.*, **107**, 624 (1957).

(⁸) W. PAULI: *Rev. Mod. Phys.*, **13**, 203 (1941).

gian only in the combination $\varphi^*\varphi$. However, this procedure destroys the geometrical significance which phase transformations have in Weyl's theory. Furthermore, in the absence of geometrical considerations, there is no *a priori* reason why the theory should be invariant under phase transformations — the Lagrangian might, for instance, contain terms of the form $\varphi^2 + \varphi^{*2}$. It would be desirable, therefore, to have a geometrical theory in which bosons and fermions are placed on the same footing.

(ii) The conditions imposed on the spin connexion do not determine it uniquely in terms of the metric. The electromagnetic potential is then the part of the spin connexion that cannot be expressed in terms of the metric. While this is a self-consistent structure, it is very unnatural from the geometric point of view. The electromagnetic potential is, as it were, thrust into the space from the outside, rather than being part of its intrinsic structure. It would be preferable if the electromagnetic potential could be expressed directly in terms of the metric (9).

(iii) The second advantage actually goes too far, for it places no restriction on the possible values of the charge of different types of particle, whereas in fact it appears that all charges are integer multiples of one basic charge.

(iv) There is an arbitrary constant multiplying the Lagrangian of the electromagnetic field, so that the unit in which the potential is measured is undetermined.

The theory described in this paper retains the advantages of Weyl's theory, but eliminates disadvantages (i), (ii) and (for bosons) (iii). If the theory could be extended to eliminate (iv), it would lead to a value for the bare fine-structure constant.

The theory will be described in detail in Sect. **3**, and its physical consequences in Sect. **4**. It can be simply summarized here by saying that it is essentially a complex version of Weyl's theory. Thus, where Weyl introduces at each point of space four linearly independent real vectors $e(\alpha)$, these vectors are assumed to be complex in our theory. This implies that the metric tensor of the space is no longer the real symmetric tensor of Riemannian geometry, but it may be taken to be either complex and Hermitian or real and non-symmetric. This is just the type of geometry used by EINSTEIN (10-12) and SCHRÖDINGER (11,12) in their unified field theories.

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(9) Cf. H. WEYL: *Proc. Nat. Acad. Sci.*, **15**, 323 (1929), bottom of p. 329.
(10) A. EINSTEIN: *Rev. Mod. Phys.*, **20**, 35 (1948).
(11) E. SCHRÖDINGER: *Space-Time Structure* (Cambridge, 1950).
(12) M.-A. TONNELAT: *La Théorie du Champ Unifié d'Einstein* (Paris, 1955).

However, the complex vector fields are needed in order both to identify the electromagnetic field and to introduce its interaction with matter.

2. – Weyl's theory.

2·1. *Gravitation.* – The starting-point of this theory is the introduction of local Cartesian axes ([13]) at each point of a Riemannian space (with Minkowskian signature).

These axes consist of four linearly independent real ([14]) vector fields $\boldsymbol{e}(\alpha)$, which I propose to call eons ([15]). When an arbitrary co-ordinate system is introduced into the space, the eons will have components $e_i(\alpha)$ ([16]). It is convenient to choose the eons so that they are (Minkowski-) orthonormal with respect to the metric g_{ij} of the Riemannian space. When this is done, we have the relation

$$g_{ij} = \eta(\alpha\beta)\, e_i(\alpha)\, e_j(\beta)\,, \tag{2.1}$$

where

$$\eta(\alpha\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Tensor quantities can then be referred to their eon-components, e.g. for a vector

$$A(\alpha) = e_i(\alpha) A^i\,.$$

The elements of interval $\mathrm{d}s^2$ can then be written

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\,,$$

$$= \eta(\alpha\beta)\,\mathrm{d}x(\alpha)\,\mathrm{d}x(\beta)\,,$$

([13]) L. P. EISENHART: *Riemannian Geometry* (Princeton, 1926), Chap. III.

([14]) WEYL takes $\boldsymbol{e}(4)$ to be pure imaginary, but as a preparation for our generalization it is more convenient to take $\boldsymbol{e}(4)$ real, and to introduce the matrix $\eta(\alpha\beta)$ in (2.1).

([15]) They are variously known as vierbeine, tetrads, tetrapods, orthopods, quadruplets, four-legs and répères mobiles.

([16]) In what follows, Greek indices always number the eons, Latin indices refer to the co-ordinate system. The summation convention is used for both sets of indices.

which shows that the $đx(\alpha)$ are co-ordinate differences in the local tangent flat space. They are written with a line through the d to emphasize that they are not perfect differentials unless the Riemannian space is flat.

In a space with a given metric, the eon field is not uniquely determined by (2.1). Any solution of (2.1) remains a solution under a Lorentz transformation $L(\alpha\beta)$ defined by ([17])

$$L'(\alpha\beta)\eta(\beta\gamma)L(\gamma\delta) = \eta(\alpha\delta)\,,$$

where L' is the transpose of L.

Furthermore, the parameters of L need not be the same at different points of the Riemannian space, but can vary arbitrarily with position.

The laws of physics are assumed to be invariant under these arbitrary Lorentz transformations ([18]) as well as under arbitrary co-ordinate transformations.

This new invariance property leads as usual to an identity. To obtain this identity, consider for simplicity the Lagrangian density $\mathcal{L}$ of a single material field, ψ. For an infinitesimal Lorentz transformation we have the variational equation

$$\delta\int \mathcal{L}\,\mathrm{d}\tau = \int \mathfrak{T}_p(\alpha)\,\delta e^p(\alpha)\,\mathrm{d}\tau + \int \frac{\delta\mathcal{L}}{\delta\psi}\,\delta\psi\,\mathrm{d}\tau = 0\,,$$

where $\mathfrak{T}_p(\alpha)\ (= \delta\mathcal{L}/\delta e^p(\alpha))$ is the energy-momentum tensor density of the material field. Assuming the material field equations $\delta\mathcal{L}/\delta\psi = 0$ to hold, we get

$$\int \mathfrak{T}_p(\alpha)\,\delta e^p(\alpha)\,\mathrm{d}\tau = 0\,.$$

Now for an infinitesimal Lorentz transformation we have

$$\delta e^p(\alpha) = \mathrm{d}o(\alpha\beta)\eta(\beta\gamma)e^p(\gamma)\,,$$

where $\mathrm{d}o(\alpha\beta)$ is an infinitesimal skew-matrix depending arbitrarily on position. It follows that $\mathfrak{T}_p(\alpha)\eta(\beta\gamma)e^p(\gamma)$, that is, $\mathfrak{T}(\alpha\beta)$, is symmetric in α, β. Since we have

$$\mathfrak{T}_{ij} = e_i(\alpha)e_j(\beta)\,\mathfrak{T}(\alpha\beta)\,,$$

it also follows that $\mathfrak{T}_{ij}$ is symmetric in i, j. In the limit of special relativity,

([17]) F. D. Murnaghan: *The Theory of Group Representations* (Baltimore, 1938), p. 352.

([18]) For our present purpose we need consider only proper Lorentz transformations.

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this symmetric $\mathfrak{T}_{ij}$ coincides with the Belinfante-Rosenfeld definition of the energy-momentum tensor density of a material field ([19]).

We now introduce an affine connexion for the eons. To do this we parallelly transfer the eons at one point P to a neighbouring point P' by means of the affine connexion Γ^i_{jk} of the Riemannian space. These transferred eons will in general differ infinitesimally from the local eons at P'. We shall make the simplest possible assumption about this difference, namely that it consists of an infinitesimal Lorentz transformation $L(\alpha\beta)$, where

$$L(\alpha\beta) = \delta(\alpha\beta) + \text{đ}o(\alpha\beta)\,,$$

$\text{đ}o(\alpha\beta)$ being an infinitesimal skew-matrix (and not a perfect differential, unless the space is flat). We further assume that $\text{đ}o(\alpha\beta)$ depends linearly on the displacement PP' $(=\mathrm{d}x^p)$; that is,

$$\text{đ}o(\alpha\beta) = o_p(\alpha\beta)\,\mathrm{d}x^p\,.$$

Then $o_p(\alpha\beta)$ is the (skew) connexion we are seeking.

This connexion can be expressed explicitly in terms of the eon field as follows. From the definition of $o_p(\alpha\beta)$, we have

$$\frac{\partial e^p(\alpha)}{\partial x^q} + \Gamma^p_{rq}e^r(\alpha) + o_q(\alpha\beta)e^p(\beta) = 0\,,$$

that is, $e^p(\alpha)$ has vanishing covariant derivative. This equation can be regarded as defining both Γ^p_{rq} and $o_q(\alpha\beta)$, for $o_q(\alpha\beta)$ can be eliminated by using its skewness in α, β, and Γ^p_{rq} can be eliminated by using its symmetry in r, q. The first elimination leads just to the Christoffel relations for Γ^p_{rq} (in terms of $e^p(\alpha)$, of course, rather than g^{pq}), while the second elimination leads to

$$\{e^q(\alpha)o_q(\beta\gamma) + e^q(\beta)\,o_q(\gamma\alpha)\} = e_p(\gamma)\left(e^q(\beta)\,\frac{\partial e^p(\alpha)}{\partial x^q} - e^q(\alpha)\,\frac{\partial e^p(\beta)}{\partial x^q}\right).$$

We can calculate $o_q(\alpha\beta)$ from this by cyclically interchanging the Greek indices and combining the resulting equations.

The gravitational field equations can now be expressed in terms of $e^p(\alpha)$ instead of g^{pq}. In order to do this we define a curvature tensor in the usual way, that is, from the change in an arbitrary vector $A(\alpha)$ when it is parallelly

([19]) L. Rosenfeld: *Acad. Roy. Belg.*, 18, No. 6 (1940).

transferred around an infinitesimal closed circuit. The result is

$$R_{pq}(\alpha\beta) = \frac{\partial o_q(\alpha\beta)}{\partial x^p} - \frac{\partial o_p(\alpha\beta)}{\partial x^q} + c_p(\alpha\gamma)\, c_q(\gamma\beta) - o_q(\alpha\gamma)\, o_p(\gamma\beta),$$

which is skew in p, q and α, β. By comparing the changes in the vectors A^i, $A(\alpha)$, we get the relation

$$R_{pq}(\alpha\beta) = R^i_{jpq}\, e_i(\alpha)\, e^j(\beta)\,, \tag{2.2}$$

where R^i_{jpq} is the Riemann-Christoffel curvature tensor.

We may define a curvature scalar R by the relation

$$R = e^p(\alpha)\, e^q(\beta)\, R_{pq}(\alpha\beta)\,.$$

It follows from (2.2) that R is equal to the Riemannian curvature scalar. The Lagrangian density of the gravitational field is now taken to be

$$\varepsilon R\,,$$

where ε is the determinant of the matrix $e_i(\alpha)$ (and is equal to $\sqrt{-g}$). The vanishing of the variation of $\int (\varepsilon R + \mathcal{L})\, d\tau$, with respect to $e^p(\alpha)$ then leads to the gravitational field equations.

2·2. *Electromagnetism.* – So far our account of Weyl's theory has consisted simply of a rewriting of Einstein's theory of gravitation in terms of eons. We must now see how Weyl introduces the electromagnetic field. This is done in terms of the spin connexion which will next be defined.

The first step is to introduce at each point of the Riemannian space Dirac matrices which are vectors with respect to the Cartesian frame $\boldsymbol{e}(\alpha)$. These matrices $\gamma(\alpha)$ satisfy the commutation relations

$$\gamma(\alpha)\gamma(\beta) + \gamma(\beta)\gamma(\alpha) = 2\eta(\alpha\beta)\mathbf{1}\,.$$

Now there exists [20] a matrix S such that

$$L(\alpha\beta)\,\gamma(\beta) = S\gamma(\alpha)S^{-1}\,. \tag{2.3}$$

This matrix S (spin transformation) is uniquely defined in terms of L up to

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[20] R. Brauer and H. Weyl: *Am. Journ. Math.*, **57**, 425 (1935); W. Pauli: *Ann. Inst. H. Poincaré*, **6**, 109 (1936).

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an arbitrary complex factor k. This factor can be normalized by requiring that

$$\det S = 1 ,$$

which restricts k to ± 1, $\pm i$. The latter two values can then be excluded for proper Lorentz transformations L by requiring that these L should give rise to a connected group of spin transformations S.

If (2.3) is solved for an infinitesimal Lorentz transformation $\delta(\alpha\beta) + đo(\alpha\beta)$, we get [21]

$$S = \mathbf{1} + \tfrac{1}{2}\, đo(\alpha\beta)\, S(\alpha\beta) ,$$

where

$$S(\alpha\beta) = \tfrac{1}{2}\{\gamma(\alpha)\gamma(\beta) - \gamma(\beta)\gamma(\alpha)\} .$$

Now in order to differentiate a spinor field with respect to position, the spinor at one point must be parallelly transferred to a neighbouring point and then subtracted from the spinor actually at that point. However, these two spinors will be referred to eons that differ by the infinitesimal Lorentz transformation $\delta(\alpha\beta) + đo(\alpha\beta)$, where $đo(\alpha\beta)$ is $o_p(\alpha\beta)\,\mathrm{d}x^p$. The spinors themselves will then differ by the corresponding infinitesimal spin transformation, in addition to their basic dependence on position. Hence the covariant derivative of a spinor field ψ is given by

$$\psi_{;p} = \frac{\partial\psi}{\partial x^p} + \frac{1}{2}\, o_p(\alpha\beta)\, S(\alpha\beta)\psi .$$

The spin connexion o_p is thus given by

$$o_p = \tfrac{1}{2}\, o_p\,(\alpha\beta)\, S(\alpha\beta)$$

If the Dirac Lagrangian is rewritten with covariant derivatives instead of ordinary derivatives, then the extra terms involving o_p describe the coupling between the ψ field and the gravitational field.

WEYL introduced the electromagnetic field into his theory by generalizing the normalization condition for the spin transformation S. Instead of restricting k by taking S to be unimodular, he allowed the phase of k to be an arbitrary function of position. This introduces an extra term into the covariant derivative of a spinor field depending on the difference $đk$ between the values of k at the two neighbouring points. This difference is assumed to depend linearly on the displacement, so we can write

$$đk = ik_p\,\mathrm{d}x^p .$$

([21]) E. M. CORSON: *Introduction to Tensors, Spinors and Relativistic Wave Equations* (London, 1953), p. 40.

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Since $\text{đ}k$ is assumed to be an imperfect differential, k_p is a new non-integrable vector field, that is, it is not in general the gradient of a scalar field. The covariant derivative is now given by

$$\psi_{;p} = \frac{\partial \psi}{\partial x^p} + c_p \psi + i k_p \psi \,,$$

so it is natural to identify k_p with the electromagnetic potential.

If this new covariant derivative is inserted into the Dirac Lagrangian, there arises the familiar coupling term

$$\mathfrak{j}^p k_p,$$

where

$$\mathfrak{j}^p = \varepsilon \psi^\dagger \gamma(\alpha) e^p(\alpha) \psi \,.$$

In this form the theory is invariant under the transformation

$$\psi' = e^{i\theta} \psi \,,$$

$$k'_p = k_p - \frac{\partial \theta}{\partial x^p} \,.$$

The identity derived from this invariance is, of course, the conservation of charge

$$\frac{\partial \mathfrak{j}^p}{\partial x^p} = 0 \,.$$

This is a covariant equation because $\mathfrak{j}^p$ is a vector density.

The scheme is completed by adding a Maxwellian Lagrangian (with an arbitrary coefficient) to the previous Lagrangian. The advantages and disadvantages of this scheme have been discussed in the introduction, where it was concluded that a new theory is needed. An attempt at such a theory is described in the next section.

3. – Theory of complex eons.

We begin by assuming that it is possible to introduce at each point of space four linearly independent *complex* vectors $\boldsymbol{e}(\alpha)$. The metric tensor of the space (g_{ij}) cannot now be real and symmetric. However, it can be either complex and Hermitian or real and non-symmetric. In the former case we can suppose that the complex eons are « orthonormal » in the sense that

$$g_{ij} = \eta(\alpha\beta) e_i(\alpha) e^*_j(\beta) \,, \tag{3.1}$$

where the star means complex conjugate. The latter case can then be derived

by taking the symmetric and skew parts of the metric to be respectively the real and imaginary parts of g_{ij}. However, we shall confine ourselves to the complex Hermitian case.

We see then that the space of the complex eon fields is also the space of the Einstein-Schrödinger theory ([10-12]). Now this theory is usually thought not to contain the electromagnetic field, since its equations of motion do not yield the Lorentz force ([22]).

This led the author ([23]) (following PAPAPETROU and ROBINSON) to suggest that the theory described just the gravitational field, which would be Hermitian (or non-symmetric) if its sources had spin.

However, we shall see that the introduction of complex eons enables one to adapt the Einstein-Schrödinger formalism so that it includes electromagnetism, without affecting the interpretation in terms of spin.

Let us first consider the physical significance of complex eons in relation to the real eons which an observer can introduce as his Cartesian reference system. In the previously quoted paper ([23]) it was shown that in the Einstein-Schrödinger theory the orbits of neutral test-particles are the geodesics of a Riemannian space whose contravariant metric tensor is $g^{\underline{ij}}$ (or, in terms of the complex metric, the real part of g^{ij}). Hence the physical space in which the observer will map the gravitational motions of bodies is this Riemannian space. The original space has no direct physical significance, but it makes its presence felt indirectly by virtue of the fact that Einstein's original (1916) field equations will not hold exactly in the Riemannian space. A similar situation arises with the complex eons. They have no direct physical meaning, but real eons which do can be constructed from them. This construction is demonstrated in the relation

$$\eta(\alpha\beta)\big(e^i(\alpha)\,e^{*j}(\beta)+e^{*i}(\alpha)\,e^j(\beta)\big)=2\Re g^{ij}=2\eta(\alpha\beta)E^i(\alpha)E^j(\beta)\,,$$

where $\Re$ means the real part of, and $E^i(\alpha)$ are the real, physical eons. Of course, the complex eons cannot be completely eliminated in this way since the real eons will not satisfy the Weyl equations described in the last section.

We can now carry through the complex generalization of the Weyl theory in the obvious way. The complex eons are not uniquely defined by (3.1); any solution remains a solution under a « quasi-unitary » transformation $U(\alpha\beta)$ defined by

$$U^{\dagger}(\alpha\beta)\,\eta(\beta\gamma)\,U(\gamma\delta)=\eta(\alpha\delta)\,,$$

where $U^{\dagger}$ is the conjugate transpose of U.

([22]) J. CALLAWAY: *Phys. Rev.*, **92**, 1567 (1953); W. B. BONNOR: *Ann. Inst. H. Poincaré*, **15**, 133 (1957).

([23]) D. W. SCIAMA: *Proc. Camb. Phil. Soc.*, **54**, 72 (1958); cf. also O. COSTA DE BEAUREGARD: *Journ. de Math.*, **22**, 85 (1943) especially footnote 1 on p. 129.

Hence the relationship (3.2) between $e^i(\alpha)$ and $E^i(\alpha)$ is unaltered by independent unitary transformations of the $e^i(\alpha)$ and Lorentz transformations of the $E^i(\alpha)$.

By analogy with the real theory of Sect. **2**, we now assume that the connexion Γ^p_{rq} is Hermitian and that the eon connexion is skew-hermitian. We write this latter connexion $u_q(\alpha\beta)$ since it has to do with quasi unitary transformations rather than Lorentz ones. The two connexions are determined by the equations

$$\frac{\partial e^p(\alpha)}{\partial x^q} + \Gamma^p_{rq} e^r(\alpha) + u_q(\alpha\beta) e^p(\beta) = 0 \,. \tag{3.3}$$

As before, the eon connexion can be eliminated, this time using its skew-hermitian property. The result is

$$\frac{\partial g^{pl}}{\partial x^q} + \Gamma^p_{rq} g^{rl} + \Gamma^l_{qr} g^{pr} = 0 \,,$$

where (3.1) has been used to eliminate the eons. This equation, with its characteristic $+-$ differentiation, is well-known from the Einstein-Schrödinger theory [10-12].

We can also eliminate Γ^p_{rq} by using its Hermitian property. This leads to

$$e^p(\gamma) e^{*q}(\alpha) u_q(\beta\gamma) + e^{*p}(\gamma) e^q(\beta) u_q(\gamma\alpha) = e^q(\beta) \frac{\partial e^{*p}(\alpha)}{\partial x^q} - e^{*q}(\alpha) \frac{\partial e^p(\beta)}{\partial x^q} \,,$$

which gives $u_q(\alpha\beta)$ in terms of the eon field.

The curvature tensor derived from $u_q(\alpha\beta)$ is given by

$$R_{pq}(\alpha\beta) = \frac{\partial u_q(\alpha\beta)}{\partial x^p} - \frac{\partial u_p(\alpha\beta)}{\partial x^q} + u_p(\alpha\gamma) u_q(\gamma\beta) - u_q(\alpha\gamma) u_p(\gamma\beta) \,, \tag{3.4}$$

which is skew in p, q and skew-hermitian in α, β. It is related to the curvature tensor formed from Γ^p_{rq} as follows

$$R_{pq}(\alpha\beta) = R^r_{spq} e_r(\alpha) e^s(\beta) \,. \tag{3.5}$$

It has two interesting contractions, the scalar curvature R given by

$$R = e^p(\alpha) e^{*q}(\beta) R_{pq}(\alpha\beta)$$

and its trace $R_{pq}(\alpha\alpha)$ (which was zero in the real case). From (3.5) we have

the relation

$$R_{rq}(\alpha\alpha) = R^{r}_{rpq}\,,$$

which we shall need later.

We now consider the physical meaning that can be given to this geometrical structure; in particular we shall attempt to identify the electromagnetic field. As was pointed out earlier, physical space is determined by the contravariant metric tensor g^{ij}. In order to describe matter, we introduce a field φ into physical space, which may transform like a scalar, spinor, vector, etc., under Lorentz transformations of the real physical eons $E^i(\alpha)$. Now comes the crucial step: we suppose in addition that φ transforms like a scalar density of weight λ under quasi-unitary transformations of the complex eons. More complicated transformations laws can be envisaged, which may have to do with families of elementary particles, but for our present purpose the discussion will be restricted to the simplest possibility.

The covariant derivative of our field quantity φ will contain an extra term arising from its density character. This is so because after parallel transfer to a neighbouring point, it will be referred to an eon frame differing from the local one by an infinitesimal quasi-unitary transformation. This transformation will alter the value of φ and so contribute to its derivative. The magnitude of this contribution is

$$\lambda u_p(\alpha\alpha)\varphi\,.$$

This suggests that we interpret $u_p(\alpha\alpha)$ as the electromagnetic potential, and λ as the charge-density of the φ field (in arbitrary units). If the φ field is quantized, λ would then be a measure of the charge of the resulting particles. We can support this interpretation by an explicit calculation of $u_p(\alpha\alpha)$ in terms of the eon field. From (3.3) we have

$$-u_q(\alpha\beta) = e_p(\beta)\,\frac{\partial e^p(\alpha)}{\partial x^q} + e_p(\beta)e^r(\alpha)\Gamma^{p}_{rq}\,.$$

Now the real part of $u_p(\alpha\beta)$ is skew in α, β, and so contributes nothing to $u_p(\alpha\alpha)$.

Hence

$$-u_q(\alpha\alpha) = i\,\mathrm{Im}\left\{e_p(\alpha)\frac{\partial e^p(\alpha)}{\partial x^q} + \Gamma^{\;p}_{pq}\right\},$$

$$= i\,\mathrm{Im}\,\frac{\partial \ln \varepsilon}{\partial x^q} + i\underset{\vee}{\Gamma^{p}_{pq}}\,.$$

It follows that a quasi-unitary transformation of the eons of determinant

$e^{i\theta}$ induces the transformations

$$\varphi' = e^{i\lambda\theta}\varphi\,,$$

$$u'_p(\alpha\alpha) = u_p(\alpha\alpha) - i\,\frac{\partial\theta}{\partial x^p}\,,$$

which is just the phase transformation characteristic of electromagnetic theory. This makes our interpretation of $u_p(\alpha\alpha)$ a reasonable one. Furthermore, by giving phase transformations this geometrical interpretation, we ensure that the φ field can interact with the electromagnetic field whether φ is a tensor or a spinor quantity in physical space. The only basic assumption we have made is that the laws of physics are invariant under quasi-unitary transformations of the complex eons. From the geometrical point of view this is a very natural assumption.

The electromagnetic field F_{pq} is defined by

$$F_{pq} = \frac{\partial u_q(\alpha\alpha)}{\partial x^p} - \frac{\partial u_p(\alpha\alpha)}{\partial x^q}\,.$$

From (3.4) we have

$$\begin{aligned} F_{pq} &= R_{pq}(\alpha\alpha)\,,\\ &= R^r_{rpq}\,,\\ &= \frac{\partial \Gamma_q}{\partial x^p} - \frac{\partial \Gamma_p}{\partial x^q}\,,\end{aligned}$$

where

$$\Gamma_q = i\Gamma^p_{pq}\,,$$

Hence the electromagnetic field in the Einstein (10) theory is R^r_{rpq}. Now if one adopts Einstein's variational principle this quantity vanishes (10), so that one would then expect the equations of motion to refer to neutral particles only. However, if one adds to the Einstein Lagrangian a Maxwellian Lagrangian for F_{pq} and a material Lagrangian (containing covariant derivatives), one obtains non-vanishing electromagnetic effects. This will be discussed in detail elsewhere. We end this paper with a brief account of some physical consequences of the theory.

4. – Physical consequences of the theory.

(*a*) From the group theoretical point of view the weight λ can be any constant. However, one can restrict its possible values by physical considerations. Suppose φ describes an assembly of bosons. This system has a

well-defined classical limit when a large number of particles are in the same state. In this limit, φ is a classical field-strength (or rather potential), which can be measured by means of a (complex) φ-charge. If this charge is g, say, then the (real) force acting on it due to φ will be [24] the gradient of the quantity $g\varphi^* + g^*\varphi$. Now if λ is not an integer, the phase of φ will not be completely determined by the field equations: φ will be of the form $\varphi_0 e^{2\pi i n\lambda}$, where n is any positive or negative integer (or zero). Similarly, g will be of the form $g_0 e^{2\pi i m\lambda}$, where m is unrelated to n. Hence the force on g will not be single-valued. However, this is impossible since the force is measurable. It follows that λ must be a positive or negative integer (or zero). Hence the charges of all bosons have (positive or negative) integer ratios.

Unfortunately, this argument does not apply to fermions, which are prevented by the Pauli exclusion principle from having a classical limit. In addition, of course, a spinor field is not measurable because it does not transform as a single-valued representation of the Lorentz group.

The quantization of charge for bosons suggests a method of calculating the (bare) fine-structure constant α. For presumably the charge e corresponds to taking λ equal to 1. However, this does not determine α directly, because at this stage of the theory the vector potential is measured in arbitrary units. These units will be determined by the Lagrangian of the theory, unless it contains an arbitrary constant. This question will be discussed in detail in another paper.

(*b*) The quasi-unitary group contains the operators that invert the eon axes $e^i(\alpha)$. Furthermore, *these operators are continuous with the identity*, for the quasi-unitary group is connected [25].

Hence, the geometrical properties of the base space (and so electromagnetic effects) cannot detect the chirality of the $E^i(\alpha)$ frame, since there is no covariant relation between the chiralities of the $e^i(\alpha)$ frame and the $E^i(\alpha)$ frame — the relation can be changed in a continuous way by quasi-unitary transformations of the $e^i(\alpha)$. It follows from this that the direct interaction between a particle and the electromagnetic field is invariant under a space reflexion and a time reflexion of the $E^i(\alpha)$.

This consequence of the theory is known to be true experimentally to considerable accuracy [3]. Until recently it would have seemed a rather trivial result, but the discovery of parity non-conservation in weak interactions [26]

[24] G. Wentzel: *Quantum Theory of Fields* (New York, 1949), p. 60.

[25] H. Weyl: *The Classical Groups* (Princeton, 1946), p. 194. Weyl proves this for the unitary group, but the proof carries through for the quasi-unitary group.

[26] C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes and R. P. Hudson: *Phys. Rev.*, **105**, 1413 (1957); R. L. Garwin, L. M. Lederman and M. Weinrich: *Phys. Rev.*, **105**, 1415 (1957); J. I. Friedman and V. L. Telegdi: *Phys. Rev.*, **105**, 1681 (1957).

has raised the general question of why parity conservation holds for some interactions and not for others.

Our theory suggests that for an interaction to be parity conserving its underlying group must in some way counteract the discontinuous nature of space-time reflexions ([27]).

Whether this is indeed true for strong (non-electromagnetic) interactions remains to be seen.

* * *

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([27]) Cf. P. A. M. DIRAC: *Rev. Mod. Phys.*, **21**, 392 (1949); J. M. JAUCH and F. ROHRLICH: *The Theory of Photons and Electrons* (Cambridge, Mass., 1955), p. 86.

RIASSUNTO (*)

Si generalizza la teoria geometrica di Weyl (1929) delle trasformazioni di gauge (fase) per fermioni e per il campo elettromagnetico in modo da renderla applicabile anche ai bosoni. Ciò si ottiene introducendo vettori *complessi*. La teoria che ne risulta è strettamente connessa alla teoria di Einstein-Schrödinger nella sua forma hermitiana e permette di identificare il tensore elettromagnetico. Dalla teoria si deducono due conseguenze fisiche: *a*) le cariche di tutti i bosoni sono multipli interi di una carica base; *b*) l'interazione diretta tra una particella e il campo elettromagnetico è invariante per le riflessioni dello spazio e del tempo. Si propone un criterio teorico (basato su un argomento che conduce a *b*)) per determinare quali interazioni conservino la parità.

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Relativistic Calculation of the Imaginary Part of Radiative Level Displacement for the Hydrogen Atom.

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Summary. — This article contains a relativistic calculation of the imaginary part of the radiative displacement of excited levels of the hydrogen atom. It starts with a quantum-electrodynamical expression for the self energy of a bound electron which has been used in the latest calculation of the Lamb shift. The imaginary part of this self energy is here considered to the same degree of accuracy as the Lamb shift. The results for the $2S$ and $2P$ levels are compared with the non-relativistic results calculated along the lines of a well known paper by Weisskopf and Wigner. It is found that there are no relativistic corrections to the desired order of accuracy.

1. – Introduction.

According to a well known paper by Weisskopf and Wigner, the interaction between the radiation field and an atom in an excited state gives rise to a displacement of the excited level, and also an exponential decrease of its probability amplitude with a life time whose reciprocal γ is equal to the total transition probability for spontaneous emission, and determines the natural line breadths. Using the idea of complex eigenvalue of Gamow and others, one may attribute the exponential decrease to an imaginary part of the radiative displacement, namely, $\operatorname{Im} \Delta E = -\hbar\gamma/2$. A non-relativistic calculation along the lines of Weisskopf and Wigner's paper gives 0 and 99.8 MHz for

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the numerical values of $\gamma/2\pi$ for the $2S$ and $2P$ levels of the hydrogen atom ([1]). The latter corresponds to an expression of $\operatorname{Im}\Delta E$ of the order of $\mu\alpha^5$, where $\mu = mc^2$ is the rest energy of the electron and $\alpha = e^2/\hbar c$ is the fine structure constant.

In recent years both the level displacement and the level width have been worked out relativistically. The level displacement calculated according to renormalized relativistic quantum-electrodynamics turns out to be in good agreement with the observed Lamb shift. On the other hand, relativistic calculations of the level width, the literature of which may be found in recent books on quantum electrodynamics ([2-4]), show that the original non-relativistic treatment of WEISSKOPF and WIGNER gives an excellent approximation.

In the present article we deal with a related problem, namely, a relativistic calculation of the imaginary part of the radiative level displacement. We take as our starting point an expression for the self energy of a bound electron derived from the Feynman-Dyson form of quantum electrodynamics. It is worked out in the Furry interaction representation, and takes account of the emission and absorption of one and only one photon. The real part of this self energy is responsible for the majority of the Lamb shift. BARENGER, BETHE and FEYNMAN ([5]) have recently improved its expectation value to the order of $\mu\alpha^6$ and have obtained a result which agrees with that of an indipendent work of KARPLUS, KLEIN and SCHWINGER ([6]). It is our object to work out the imaginary part of the same expression to the same degree of accuracy. We do not consider the contribution from the emission and absorption of two photons, which is known to give a very small Lamb shift ([7]).

The fact that the imaginary part of this self energy when calculated does indeed give the correct rate of radiation was briefly mentioned by FEYNMAN. Using the relation between $\operatorname{Im}\Delta E$ and the transition rate γ one may derive the dependence of $\operatorname{Im}\Delta E$ on the matrix elements of radiative transitions from that o γ. For the sake of completeness we shall give in Sect. **2** a direct derivation of a formula for $\operatorname{Im}\Delta E$ which will be needed for subsequent considerations. Our derivation is the exact relativistic version of a derivation of reference ([3]) for the Weisskopf-Wigner approximation.

On the basis of the formula derived in Sect. **2**, we shall discuss in Sect. **3** the numerical values of $\operatorname{Im}\Delta E$ for the excited $2S$ and $2P$ levels of the hydrogen

([1]) W. E. LAMB: *Reports on Progress in Physics*, **14**, 19 (1951).

([2]) W. HEITLER: *Quantum Theory of Radiation* (Oxford, 1954).

([3]) J. M. JAUCH and F. ROHRLICH: *Theory of Photons and Electrons* (Cambridge Mass., 1955).

([4]) H. UMEZAWA: *Quantum Field Theory* (Amsterdam, 1956).

([5]) M. BARANGER, H. A. BETHE and R. P. FEYNMAN: *Phys. Rev.*, **92**, 482 (1953).

([6]) R. KARPLUS, A. KLEIN and J. SCHWINGER: *Phys. Rev.*, **86**, 288 (1952).

([7]) E. E. SALPETER: *Phys. Rev.*, **89**, 92 (1953).

atom. There is no difficulty in working out the exact results, but it will be sufficient for practical purpose to assess the numerical values to the order of $\mu\varkappa^6$. It will be seen that, to our prescribed approximation, there are no relativistic corrections to the numerical values mentioned at the beginning of this section.

2. – Derivation of $\mathrm{Im}\,\Delta E$ from quantum electrodynamics.

In the following we put c and $\hbar$ equal to unity and use the same $\boldsymbol{\alpha}$, β matrices as in reference (2). The Dirac equation for a stationary state n of an electron in the Coulomb field of a proton is then of the form

$$[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m+V(r)]\psi_n(\boldsymbol{x})=E_n\psi_n(\boldsymbol{x})\,, \tag{1}$$

where

$$V(r)=-e^2/r \tag{2}$$

and E_n may be positive or negative. The corresponding time-dependent wave function is

$$\psi_n(\mathfrak{x})=\psi_n(\boldsymbol{x})\exp[-iE_nt]\,. \tag{3}$$

We assume $\psi_n(\boldsymbol{x})$ to be normalized, and use $\varphi_n(\boldsymbol{p})$ to denote its Fourier transform. Let $\overline{\psi}_n=\psi_n^*\beta$, $\overline{\varphi}_n=\varphi_n^*\beta$.

We use the same γ matrices $\gamma_i=\beta\alpha_i$, $\gamma_0=\beta$, the same summation convention, and the same propagation kernel (denoted by $K(\mathfrak{x}_2,\mathfrak{x}_1)$ here) as in reference (5). The Fourier transform of $K(\mathfrak{x}_2,\mathfrak{x}_1)$ is of the form

$$K(\mathfrak{p}_2,\mathfrak{p}_1)=\delta(p_{20}-p_{10})\,K(E;\boldsymbol{p}_2,\boldsymbol{p}_1)\qquad(E=p_{10}=p_{20}). \tag{4}$$

As in reference (5), we take as our starting point the following expression for the self energy of an electron in a bound state a of a hydrogen atom:

$$\Delta E_a=-e^2(4\pi^3)^{-1}\int\overline{\psi}_a(\mathfrak{x}_2)\gamma_\mu K(\mathfrak{x}_2,\mathfrak{x}_1)\gamma_\mu\psi_a(\mathfrak{x}_1)(1/\mathfrak{k}^2)\exp[-i\mathfrak{k}\cdot(\mathfrak{x}_2-\mathfrak{x}_1)]\cdot\\ \cdot\,\mathrm{d}^4k\,\mathrm{d}^3x_2\,\mathrm{d}^3x_1\,\mathrm{d}(t_2-t_1) \tag{5}$$

where

$$\mathfrak{k}^2=\omega^2-k^2\,, \tag{6}$$

$k=|\boldsymbol{k}|$ with an infinitesimal negative imaginary part, and $\mathrm{d}^4k=\mathrm{d}^3k\,\mathrm{d}\omega$. In

terms of the Fourier transforms, ΔE_a becomes

$$\Delta E_a = -\, e^2(4\pi^3)^{-1}\int \bar{\varphi}_a(\boldsymbol{p}_2)\gamma_\mu K(E_a-\omega;\, \boldsymbol{p}_2-\boldsymbol{k},\, \boldsymbol{p}_1-\boldsymbol{k})\cdot \gamma_\mu \varphi_a(\boldsymbol{p}_1)(1/\mathfrak{k}^2)\,\mathrm{d}^4k\,\mathrm{d}^3p_2\,\mathrm{d}^3p_1\,. \tag{7}$$

According to the procedure of mass renormalization, a term involving the self mass Δm should be added to ΔE_a. Since this renormalization term is real and we are interested only in the imaginary part, it is left out for brevity.

To separate out the imaginary part of ΔE_a we use the following general expression for the propagation kernel

$$K(\mathfrak{x}_2, \mathfrak{x}_1) = \begin{cases} \displaystyle\sum_{E_n>0} \psi_n(\mathfrak{x}_2)\,\bar{\psi}_n(\mathfrak{x})_1 & (t_2 > t_1), \\ \displaystyle -\sum_{E_n<0} \psi_n(\mathfrak{x}_2)\,\bar{\psi}_n(\mathfrak{x}_1) & (t_2 < t_1). \end{cases} \tag{8}$$

Eqs. (4) and (8) give rise to

$$K(E;\, \boldsymbol{p}_2, \boldsymbol{p}_1) = i\sum_n \varphi_n(\boldsymbol{p}_2)\bar{\varphi}_n(\boldsymbol{p}_1)/(E-E_n)\,, \tag{9}$$

where E_n contains an infinitesimal imaginary part which is negative for $E_n > 0$ and positive for $E_n < 0$. Using the formula

$$\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{1}{\omega^2-k^2}\,\frac{1}{\omega-(E_a-E_b)} = \frac{\pi i}{k}\,\frac{1}{E_a-E_b-\sigma_b k}\,, \quad (\sigma_b = E_b/|E_b|), \tag{10}$$

we obtain from eqs. (7) and (9)

$$\Delta E_a = -\, e^2(4\pi^2)^{-1}\sum_b \int \mathrm{d}^3k\, \Gamma^{ab}_\mu(\boldsymbol{k})\Gamma^{ab}_\mu(\boldsymbol{k})^*[k(E_a-E_b-\sigma_b k)]^{-1}\,, \tag{11}$$

where

$$\Gamma^{ab}_\mu(\boldsymbol{k}) = \int \bar{\varphi}_a(\boldsymbol{p})\gamma_\mu \varphi_b(\boldsymbol{p}-\boldsymbol{k})\,\mathrm{d}^3p\,. \tag{12}$$

An alternative espression for $\Gamma^{ab}_\mu(\boldsymbol{k})$ that follows from eq. (12) is

$$\Gamma^{ab}_\mu(\boldsymbol{k}) = \int \bar{\psi}_a(\boldsymbol{x})\gamma_\mu \psi_b(\boldsymbol{x})\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]\,\mathrm{d}^3x\,. \tag{13}$$

Owing to the infinitesimal imaginary parts of E_b and k,

$$\frac{1}{E_a-E_b-\sigma_b k} = \frac{P}{E_a-E_b-\sigma_b k} - \pi i\sigma_b\delta(E_a-E_b-\sigma_b k)\,, \tag{14}$$

where P denotes the principal value and the expression $E_a - E_b - \sigma_b k$ is taken to be real on the right-hand side. The term involving the δ-function gives rise to an imaginary part of ΔE_a, namely

$$\text{(15)} \qquad \operatorname{Im} \Delta E_a = e^2(4\pi)^{-1} \sum_b (E_a - E_b) \int \mathrm{d}\Omega \Gamma_\mu^{ab}(\boldsymbol{k}) \Gamma_\mu^{ab}(\boldsymbol{k})^* , \quad (k = E_a - E_b) ,$$

where $\sum_b$ extends over all the positive-energy states b with $E_b < E_a$, and $\mathrm{d}\Omega$ denotes an element of solid angle in the $\boldsymbol{k}$-space.

The right-hand side of eq. (15) involves in a symmetrical way all the four components of the photon field. Since only transverse photons are emitted in a physical process, it is to be expected that the longitudinal and scalar components cancel each other. That this is indeed the case may be seen as follows. From eq. (1) we obtain, for $k = E_a - E_b$,

$$\text{(16)} \qquad \int \psi_a^*(\boldsymbol{x})(k - \boldsymbol{\alpha} \cdot \boldsymbol{k}) \psi_b(\boldsymbol{x}) \exp[i\boldsymbol{k} \cdot \boldsymbol{x}] \, \mathrm{d}^3 x = 0 .$$

Hence

$$\text{(17)} \qquad \Gamma_0^{ab}(\boldsymbol{k}) = (1/k) \boldsymbol{k} \cdot \boldsymbol{\Gamma}^{ab}(\boldsymbol{k}) ,$$

and therefore eq. (15) reduces to

$$\text{(18)} \qquad \operatorname{Im} \Delta E_a = e^2(4\pi)^{-1} \sum_b (E_b - E_a) \int \mathrm{d}\Omega \sum_{\boldsymbol{e}} |\boldsymbol{e} \cdot \boldsymbol{\Gamma}^{ab}(\boldsymbol{k})|^2 , \quad (k = E_a - E_b),$$

where $\boldsymbol{e}$ denotes a unit vector of polarization. The result now depends only on the transverse component of the vector

$$\text{(19)} \qquad \boldsymbol{\Gamma}^{ab}(\boldsymbol{k}) = \int \psi_a^*(\boldsymbol{x}) \boldsymbol{\alpha} \psi_b(\boldsymbol{x}) \exp[i\boldsymbol{k} \cdot \boldsymbol{x}] \, \mathrm{d}^3 x .$$

Eq. (18) gives the exact relation between $\operatorname{Im} \Delta E_a$ and the relativistic transition matrix-elements associated with the emission of transverse photons. The usual non-relativistic result may be derived from it by making two approximations: (i) the dipole approximation which amounts to putting the exponential factor equal to unity; (ii) the non-relativistic approximation, i.e. the use of the Schrödinger wave functions, χ say, instead of the Dirac wave functions ψ. Making these approximations in eqs. (18), (19) one obtains

$$\text{(20)} \qquad \operatorname{Im} \Delta E_a = (2/3) e^2 \sum_b (E_b - E_a) |\boldsymbol{\Gamma}^{ab}|^2 ,$$

with

$$\text{(21)} \qquad \boldsymbol{\Gamma}^{ab} = (1/m) \int \chi_a^*(\boldsymbol{x}) \boldsymbol{p} \chi_b(\boldsymbol{x}) \, \mathrm{d}^3 x .$$

3. – Numerical values of $\mathrm{Im}\,\Delta E$

Using the non-relativistic Schrödinger wave functions for the hydrogen atom, one obtains from eq. (21) the results

$$|\boldsymbol{\Gamma}^{ab}|^2 = 0 \qquad (2S \to 1S), \tag{22}$$

$$|\boldsymbol{\Gamma}^{ab}|^2 = 2(2/3)^8\alpha^2 \qquad (2P \to 1S). \tag{23}$$

Eqs. (20), (22), (23) and the Bohr formula

$$E_a - E_b = (3/8)m\alpha^2 \tag{24}$$

lead the non-relativistic results mentioned in the introduction, namely

$$\mathrm{Im}\,\Delta E|_{2S} = 0\,, \tag{25}$$

$$\mathrm{Im}\,\Delta E|_{2P} = -\,(1/2)(2/3)^8\, m\alpha^5\,. \tag{26}$$

It is our object to consider the realtivistic corrections to eqs. (25) and (26). The evaluation of the integral in eq. (19) requires the use of the Dirac wave functions for the hydrogen atom. Let $Q^{(l)}_{j,m}$ denote the normalized spherical-harmonic spinor associated with the quantum numbers l, j, m [8]. For example,

$$Q^{(0)}_{\frac{1}{2},\frac{1}{2}} = (4\pi)^{-\frac{1}{2}}\begin{pmatrix}1\\0\end{pmatrix}, \qquad Q^{(0)}_{\frac{1}{2},-\frac{1}{2}} = (4\pi)^{-\frac{1}{2}}\begin{pmatrix}0\\1\end{pmatrix}, \tag{27}$$

$$Q^{(1)}_{\frac{1}{2},\frac{1}{2}} = (4\pi)^{-\frac{1}{2}}(1/r)\begin{pmatrix}z\\x+iy\end{pmatrix}, \tag{28}$$

$$Q^{(0)}_{\frac{3}{2},\frac{3}{2}} = -\,(4\pi)^{-\frac{1}{2}}(\tfrac{3}{2})^{\frac{1}{2}}(1/r)\begin{pmatrix}x+iy\\0\end{pmatrix}. \tag{29}$$

The large and small components of the Dirac wave functions for the states under consideration are then of the following form

$$\psi_1 = (1/r)\,u_1(r)\,Q^{(0)}_{\frac{1}{2},m} \qquad \psi_2 = (1/r)\,u_2(r)\,Q^{(1)}_{\frac{1}{2},m} \qquad (2S_{\frac{1}{2}}) \tag{30}$$

$$\psi_1 = (1/r)\,u_1'(r)\,Q^{(1)}_{\frac{1}{2},m} \qquad \psi_2 = (1/r)\,u_2'(r)\,Q^{(0)}_{\frac{1}{2},m} \qquad (2P_{\frac{1}{2}}) \tag{31}$$

$$\psi_1 = (1/r)\,u_1''(r)\,Q^{(1)}_{\frac{3}{2},m} \qquad \psi_2 = (1/r)\,u_2''(r)\,Q^{(2)}_{\frac{3}{2},m} \qquad (2P_{\frac{3}{2}}) \tag{32}$$

$$\psi_1 = (1/r)\,v_1(r)\,Q^{(0)}_{\frac{1}{2},m} \qquad \psi_2 = (1/r)\,v_2(r)\,Q^{(1)}_{\frac{1}{2},m} \qquad (1S_{\frac{1}{2}}) \tag{33}$$

[8] H. A. KRAMERS: *Quantum Mechanics* (Amsterdam, 1957).

On account of the factor $E_a - E_b$ in eq. (18), the transitions between the states of the same n are negligible, so that the $1S$ states are the only final states to be taken into account. Consider first the transitions $2S_{\frac{1}{2}} \to 1S_{\frac{1}{2}}$: From eq. (19), (30), (33), (27), (28) we obtain, for the initial state $m_a = \frac{1}{2}$ and final states $m_b = \pm\frac{1}{2}$,

$$\text{(34)} \qquad \begin{cases} (\boldsymbol{\Gamma}\cdot\boldsymbol{e})_{\frac{1}{2}\frac{1}{2}} = I(e_1k_2 - e_2k_1)(1/k) \\ (\Gamma\cdot\boldsymbol{e})_{\frac{1}{2},-\frac{1}{2}} = -iI[e_3(k_1 - ik_2) - (e_1 - ie_2)k_3](1/k)\,, \end{cases}$$

where

$$\text{(35)} \qquad I = \int_0^\infty (u_2^* v_1 - u_1^* v_2) j_1(kr)\,\mathrm{d}r\,,$$

j_1 being a spherical Bessel function. Summing over the two directions of polarization $\boldsymbol{e}$ and integrating over all direction of $\boldsymbol{k}$ we obtain from eq. (34)

$$\text{(36)} \qquad \int \mathrm{d}\Omega \sum_{m_b} \sum_{\boldsymbol{e}} |(\boldsymbol{\Gamma}\cdot\boldsymbol{e})_{m_a.m_b}|^2 = 8\pi |I|^2\,.$$

It can be shown that the same result holds also for the initial state $m_a = -\frac{1}{2}$.

The normalized radial wave functions u, v involved in the integral of eq. (35) may conveniently be expressed in terms of the variable

$$\text{(37)} \qquad \varrho = m\alpha r\,.$$

We have, approximately,

$$\text{(38)} \qquad \begin{cases} u_1 = -(m\alpha/2)^{\frac{1}{2}} \exp[-\varrho/2]\varrho(1 - \varrho/2) \\ u_2 = -i\alpha(m\alpha/8)^{\frac{1}{2}} \exp[-\varrho/2]\varrho(1 - \varrho/4) \end{cases}$$

$$\text{(39)} \qquad \begin{cases} v_1 = -2(m\alpha)^{\frac{1}{2}} \exp[-\varrho]\varrho \\ v_2 = -i\alpha(m\alpha)^{\frac{1}{2}} \exp[-\varrho]\varrho\,. \end{cases}$$

The exact expressions differ from these by factors which are functions of ϱ and α^2, and which tend to unity as $\alpha^2 \to 0$.

The function j_1 may be expanded into a power series of the form

$$\text{(40)} \qquad j_1(kr) = (1/3)kr + \ldots\,.$$

On account of eq. (37), the variable r is of the order of $1/m\alpha$ relative to the

variable ϱ. Hence the factor kr is of the order of α relative to ϱ. As mentioned in the Introduction, we wish to evaluate $\operatorname{Im}\Delta E_a$ to the order $m\alpha^6$. This requires the determination of I to the order α^2. One may therefore take only the leading term in the expansion of eq. (40), so that eq. (35) becomes

$$I = (k/3)\int_0^\infty (u_2^* v_1 - u_1^* v_2) r\,\mathrm{d}r\,, \tag{41}$$

and use the approximate expressions (38), (39). A simple calculation shows that the integral I then vanishes. Hence there is no relativistic correction to the result (25) for the $2S$ states, in our approximation.

Alternatively, one may attain the same degree of accuracy by expanding the exponential function $\exp[i\boldsymbol{k}\cdot\boldsymbol{x}]$ in eq. (19) and take only the first two terms, which correspond to the dipole and quadrupole radiation. This procedure leads again to eqs. (34), with I given by the approximate expression (41).

Let us consider now the transitions $2P \to 1S$ to the same order of accuracy. Again, we may confine our attention to the dipole and quadrupole radiation. It can be seen from eqs. (31)-(33) that there is no quadrupole radiation for $2P \to 1S$. One may therefore omit the exponential factor altogether. It then follows that, for the transitions $2P_{\frac{1}{2}} \to 1S_{\frac{1}{2}}$ $(m_a = \frac{1}{2},\ m_b = \pm\frac{1}{2})$,

$$\begin{cases} (\boldsymbol{\Gamma}\cdot\boldsymbol{e})_{\frac{1}{2},\frac{1}{2}} = I' e_3\,, \\ (\boldsymbol{\Gamma}\cdot\boldsymbol{e})_{\frac{1}{2},-\frac{1}{2}} = I'(e_1 - i e_2)\,, \end{cases} \tag{42}$$

where

$$I' = \int_0^\infty \left(u_2'^* v_1 - \frac{1}{3}\, u_1'^* v_2\right) \mathrm{d}r\,. \tag{43}$$

From eqs. (42),

$$\int \mathrm{d}\Omega \sum_{m_b} \sum_{\boldsymbol{e}} |(\boldsymbol{\Gamma}\cdot\boldsymbol{e})_{m_a,m_b}|^2 = 8\pi |I'|^2\,, \tag{44}$$

a result which is valid also for $m_a = -\frac{1}{2}$. Now v_1, v_2 are given by the approximate expressions (39) and u_1', u_2' by

$$\begin{cases} u_1' = (m\alpha/24)^{\frac{1}{2}} \exp[-\varrho/2]\,\varrho^2\,, \\ u_2' = -i\alpha(3m\alpha/32)^{\frac{1}{2}} \exp[-\varrho/2]\,\varrho(1-\varrho/6)\,, \end{cases} \tag{45}$$

which differ from the exact ones by factors of a type mentioned previously.

Consequently, the value of I' to the order of α^2 may be obtained by using eqs. (39), (45), i.e.

$$I' = -i\alpha(2/3)^{\frac{9}{2}} \, . \tag{46}$$

The result that follows from eqs. (18), (44), (46) turns out to be identical with the non-relativistic result (26). A similar calculation shows that this is also true of an initial $2P_{\frac{3}{2}}$ state. We reach therefore the conclusion that there is no relativistic correction for the $2P$ states to the desired order of accuracy, just as in the case of the $2S$ states.

The agreement of the relativistic and non-relativistic calculations is not accidental. The explanation hinges on the fact that a calculation of Γ^{ab} using the Dirac radial wave-functions in the approximate form of eqs. (38), (39), (45) is equivalent to a calculation using the Schrödinger radial wave-functions.

* * *

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RIASSUNTO (*)

Il presente articolo presenta un calcolo relativistico della parte immaginaria dello spostamento radiativo dei livelli eccitati dell'atomo di idrogeno. Il calcolo parte da un'espressione elettrodinamica quantistica per la self-energia di un elettrone legato usata nel più recente calcolo dello spostamento di Lamb. La parte immaginaria di questa self-energia è considerata nel lavoro con lo stesso grado di approssimazione dello spostamento di Lamb. Si confrontano i risultati per i livelli $2S$ e $2P$ coi risultati non relativistici calcolati secondo lo schema di un ben noto lavoro di WEISSKOPF e WIGNER. Si trova che l'ordine d'approssimazione desiderato non richiede correzioni relativistiche.

(*) *Traduzione a cura della Redazione.*

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Eine Tabelle für den Factor der Inkohärenten Streuung.

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Zusammenfassung. — In dieser Arbeit formen wir den Faktor der Inkohärenten Streuung in eine solche Form um, welche uns die Möglichkeit einer einfachen praktischen Ausrechnung gibt. Mit Hilfe der sehr genauen numerischen Thomas-Fermi-Werten von Kobayashi wird der Faktor der Inkohärenten Streuung tabellarisiert. In dieser Arbeit leiten wir auch eine Approximation für den Faktor der Inkohärenten Streuung ab.

Im Anschluß an die WALERSCHE (1) Streuformel konnte HEISENBERG (2) zeigen, daß die Intensität gestreuter Röntgenstrahlen durch den $s_0^2(w)$ Faktor bestimmt ist. Der Faktor $s_0^2(w)$ hat folgende Gestalt

$$s_0^2 = 1 \quad \int_0^{x_0} \left[\left(\frac{\varphi_0(x)}{x}\right)^{\frac{1}{2}} - w\right]^2 \left[\left(\frac{\varphi_0(x)}{x}\right)^{\frac{1}{2}} + \frac{1}{2}\, w\right] x^2 \,\mathrm{d}x\,, \tag{1}$$

wo $\varphi_0(x)$ die Thomas-Fermi Funktion des freien neutralen Atoms bezeichnet und x_0 ist die Wurzel der Gleichung (3)

$$\left(\frac{\varphi_0(x_0)}{x_0}\right)^{\frac{1}{2}} = w = \frac{0.176\cdot 10^{-8}\ \mathrm{cm}}{Z^{\frac{2}{3}}}\,\frac{4\pi}{\lambda}\sin\frac{\vartheta}{2}\,. \tag{2}$$

Eine leichte Rechnung zeigt, daß man den Faktor $s_0^2(w)$ mit Hilfe der Thomas-Fermischen Differentialgleichung $\varphi_0'' = \varphi_0^{\frac{3}{2}} x^{\frac{1}{2}}$ und der Randbedingungen

(1) I. WALLER und D. R. HARTREE: *Proc. Roy. Soc. London*, A **124**, 119 (1929).

(2) W. HEISENBERG: *Zeits. f. Phys.*, **83**, 555 (1932).

(3) R. GOMBÁS: *Die statistische Theorie des Atoms und ihre Anwendungen* (Wien, 1949), p. 249. Vgl. hierzu auch H. KOPPE: *Zeits. f. Phys.*, **124**, 658 (1948).

Tabelle I. — *Der Faktor* $s_0^2(w)$.

x_0	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.9	0.9
w	∞	2.96934	1.99131	1.54988	1.28408	1.10181	0.967092	0.862546	0.778564	0.70935
s_0^2	1	0.9973	0.9920	0.9854	0.9783	0.9699	0.9608	0.9512	0.9412	0.9308

x_0	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8
w	0.65116	0.55845	0.487633	0.431639	0.386199	0.348576	0.316910	0.289899	0.266603	0.246316
s_0^2	0.9202	0.8984	0.8764	0.8542	0.8322	0.8105	0.7892	0.7684	0.7481	0.7284

x_0	3.0	3.2	3.4	3.6	3.8	4.0	5.0	6.0	7.0	8.0
w	0.228495	0.212735	0.198709	0.186145	0.174846	0.16461	0.125545	0.099518	0.0811507	0.0676267
s_0^2	0.7092	0.6906	0.6726	0.6553	0.6384	0.6221	0.5489	0.4869	0.4347	0.3904

x_0	9.0	10.0	11.0	12.0	13.0	14.0	17.0	20.0	26.0	∞
w	0.0573401	0.0493092	0.0429058	0.0377094	0.0334273	0.029854	0.0220598	0.0170072	0.0110403	0.0
s_0^2	0.3525	0.3187	0.2916	0.2669	0.2449	0.2262	0.1808	0.1479	0.09996	0.0

für die Thomas-Fermi Funktion $\varphi_0(0) = 1$ und $\varphi_0'(0) = \text{Const}$ in folgender Gestalt schreiben kann,

$$s_0^2 = \varphi_0(x_0) - x_0\varphi_0'(x_0) - \frac{(x_0\varphi_0(x_0))^3}{6} + \frac{3w}{2}\int_0^{x_0} x\varphi_0(x)\,\mathrm{d}x\,. \tag{3}$$

Die letzte Gleichung erlaubt uns mit Hilfe der sehr genauen $\varphi_0(x)$-Werte und $\varphi_0'(x)$-Werte von KOBAYASHI (4) den Faktor $s_0^2(w)$ numerisch zu berechnen. Das Integral welches in der Formel (3) vorkommt, haben wir mit Hilfe der der Simpsonschen Formel berechnet. In Tabelle I haben wir die Werte des Faktors $s_0^2(w)$ für verschiedene x_0 oder w-Werte.

BEWILOGUA (5) und LENZ (6) haben den Faktor $s_0^2(w)$ für einige w-Werte numerisch berechnet, Unsere Tabelle I ergänzt systematisch die numerischen $s_0^2(w)$-Werte von BEWILOGUA. Eine solche oben angeführte Ergänzung der $s_0^2(w)$-Werte von BEWILOGUA ist nur möglich mit Hilfe der sehr genauen Werte für $\varphi_0(x)$ und $\varphi_0'(x)$ von KOBAYASHI und unserer Formel (3). Setzt man für $\varphi_0(x)$ die Approximation der Thomas-Fermi Funktion des Autors (7),

$$\varphi_0(x) = \frac{1}{(1 + ax + bx^2)^{\frac{3}{2}}} \quad \text{mit} \quad \begin{aligned} a &= 0.710\,5\,, \\ b &= 0.039\,19\,. \end{aligned} \tag{4}$$

so bekommen wir für den Faktor $s_0^2(w)$ in unserem Falle folgende Formel für $s_0^2(w)$ nämlich:

$$s_0^2 = \frac{2 + 5ax_0 + 8bx_0^2}{2(1 + ax_0 + bx_0^2)^{\frac{3}{2}}} + \frac{3(2 + ax_0)w}{(a^2 - 4b)(1 + ax_0 + bx_0^2)^{\frac{1}{2}}} - \frac{6w}{(a^2 - 4b)} - \frac{(wx_0)^3}{6}\,. \tag{5}$$

TABELLE II. – *Ein Vergleich für* $s_0^2(w)$.

Numerische Werte von BEVILOGUA	w	0.0	0.025	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.8	∞
	s_0^2	0	0.199	0.319	0.486	0.674	0.776	0.839	0.880	0.909	0.944	1
Unsere Werte Gl. (5)	w	0.0	0.025	0.05	0.1	0.2	0.3	0.4	0.5	0.	0.8	∞
	x_0	∞	15.701	9.858	5.924	3.349	2.310	1.735	1.368	1.113	0.7851	0.0
	s_0^2	0	0.196	0.317	0.480	0.671	0.783	0.856	0.906	0.940	0.982	1

(4) S. KOBAYASHI: *unter Druck.* Vgl. P. GOMBÁS: *Handb. der Phys.* Band XXXVI (Wien, 1956), p. 127.

(5) L. BEWILOGUA: *Phys. Zeits.*, **32**, 740 (1931).

(6) F. LENZ: *Zeits. f. Phys.*, **135**, 248 (1953).

(7) T. TIETZ: *Nuovo Cimento*, **4**, 1192 (1956).

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In Tabelle II haben wir einen Vergleich für den Faktor $s_0^2(w)$ in unserem Falle mit den numerischen Werten für $s_0^2(w)$ von Bewilogua. Tabelle II zeigt, daß für $x_0 \geqslant 0.8$ bzw. $w < 0.7$ die Anpassung unserer $s_0^2(w)$-Werte an die numerisch berechneten $s_0^2(w)$-Werte gut ist. In einer späteren Arbeit werden wir andere Approximationen für $s_0^2(w)$ angeben.

RIASSUNTO (*)

Nel presente lavoro trasformiamo il fattore della dispersione incoerente in una forma che ci dà la possibilità di calcolarlo praticamente in modo semplice. Con l'ausilio degli esattissimi valori numerici di Thomas-Fermi dati da Kobayashi si tabula il fattore della dispersione incoerente. Nel presente lavoro deriviamo anche un'approssimazione per il fattore della dispersione incoerente.

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Gamma-Gamma Angular Correlations with Circular Polarization Measurements.

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(ricevuto il 26 Gennaio 1958)

Summary. — The problem of the γ-γ double angular correlation is examined in which the circular polarization state of photons is observed. The relation among the various possible correlations is discussed. The Legendre polynomials of odd order introduce an asymmetry around the value $\beta = \pi/2$, where β is the angle between the two radiations. Numerical calculations have been made for the γ-γ cascade of ^{60}Co.

1 – Introduction.

The measurement of the circular polarization of γ-rays can be considered already a common experimental technique. We have therefore considered it useful to examine the problem of the γ-γ angular correlations in which one observes the circular polarization state of the photons and which we have found to have been not yet examined in detail.

Notation and formalism used are those taken from the work of Biedenharn and Rose [1] (referred to hereafter as BR), and, for simplicity, we shall treat only the case of two successive γ-radiations in which the initial and final nuclear states are randomly oriented and the intermediate state is unperturbed. It is assumed that the parity and the angular momenta of the nuclear states are well defined. In the first time we shall not assume that the two radiations are pure multipoles.

(1) L. C. Biedenharn and M. E. Rose: *Rev. Mod. Phys.*, **25**, 729 (1953).

Under the above assumptions, the general double correlation function can be written as follows:

$$(1)\qquad W(\alpha\beta\gamma) \sim \sum (-1)^{L_1+L_2+\nu} c_{\nu\tau_1}(L_1L_1')\, c_{\nu\tau_2}(L_2L_2')(j_1\|L_1\|j)(j_1\|L_1'\|j)(j_2\|L_2\|j)(j_2\|L_2'\|j)\cdot$$
$$\cdot W(jjL_1L_1';\nu j_1)\, W(jjL_2L_2';\nu j_2)\, D(\nu_1\tau_2\tau_1;\alpha\beta\gamma)\,.$$

Where the summation is over L_1, L_1', L_2, L_2', τ_1, τ_2 and ν.

α, β, γ are the Euler angles of the rotation which carries the co-ordinate system of the first radiation into the co-ordinate system of the second one.

j_1, j and j_2 are respectively the angular momentum quantum number for the initial, intermediate and final nuclear state, while L_1, L_1' and L_2, L_2' represent the angular momentum quantum numbers for the first and the second radiation.

ν is an integer number which satisfies the relation

$$0 \leqslant \nu \leqslant \mathrm{Min}\,(2j,\, 2\,|j_1-j\,|,\, 2\,|j_2-j\,|)\,;$$

τ_i is defined by $\tau_i = L_{iz}' - L_{iz}$, where L_{iz}' and L_{iz} are the projections of L_i' and L_i along the axis of motion of the i-th radiation.

$W(jjL_iL_i';\nu j_i)$ are the Racah coefficients and $D(\nu,\tau_2\tau_1;\alpha\beta\gamma)$ is the usual rotation matrix of order ν. Both are purely geometrical factors.

$(j_i\|L_i\|j)$ and $(j_i\|L_i'\|j)$ are the reduced matrix elements for the transition which goes from the state of angular momentum j_i to the state of angular momentum j through emission of pure multipoles of order L_i and, respectively, L_i'.

The coefficients $c_{\nu\tau}(LL')$ are characteristic of the radiation to which they refer, of its parity, and multipolarity, and also of the type of observations which are performed on the radiation during the experiment under discussion. So, if, for example, we observe only the direction of motion of the γ-ray, we find that (BR, Sect. III-A):

$$(2)\qquad \begin{cases} {}^{\mathrm{dir}}c_{\nu 0} = (-1)^{L+1}[(2L+1)(2L'+1)]^{\frac{1}{2}}\, C(LL'\nu;\, 1\,-1) & \text{for } \nu \text{ even,} \\ {}^{\mathrm{dir}}c_{\nu 0} = 0 & \text{for } \nu \text{ odd,} \end{cases}$$

while ${}^{\mathrm{dir}}c_{\nu\tau} = 0$ for $\tau \neq 0$.

The last property generally holds when the observation of the radiation does not depend from rotations around the direction of motion.

The calculation of the coefficients $c_{\nu 0}(LL')$ for γ-rays of which the right (RC) or the left (LC) circular polarization state is observed, can be made according to the procedure indicated in BR, Sect. II-B and Sect. III-A; the following

result is obtained:

$$(3)\qquad \begin{cases} {}^{\mathrm{LC}}c_{\nu 0} = (-1)^{L+1}\quad [(2L+1)(2L'+1)]^{\frac{1}{2}}\, C(LL'\nu;\, 1-1)\,, \\ {}^{\mathrm{RC}}c_{\nu 0} = (-1)^{L+1-\nu}[(2L+1)(2L'+1)]^{\frac{1}{2}}\, C(LL'\nu;\, 1-1)\,, \end{cases}$$

with ν any integer number.

We can note, that, for even values of ν:

$$(4)\qquad {}^{\mathrm{LC}}c_{\nu 0} = {}^{\mathrm{RC}}c_{\nu 0} = {}^{\mathrm{dir}}c_{\nu 0} \qquad (\nu \text{ even}).$$

2 – Angular correlations with circular polarization measurements.

It is intuitive, and follows immediately from the eq. (1)–(4), that the measurement of the circular polarization of one of the two quanta, together with the measurement of the direction of the other one, leads to the same correlation function that we would have obtained by simply observing the direction of both quanta.

Likewise, if we remember that the coefficients for the linearly polarized radiation vanish when ν is odd, we come to the conclusion that the measurement of the circular polarization of one of the two quanta, together with the measurement of the linear polarization of the other one, leads to the same correlation function that we would have obtained by simply substituting the circular polarization measurement with the measurement of the direction.

We have still to examine the case in which the circular polarization state of both γ-rays is observed.

Eq. (1), considering that $\tau_1 = \tau_2 = 0$, that ${}^{\mathrm{LC}}c_{\nu 0}$ and ${}^{\mathrm{RC}}c_{\nu 0}$ are real, that $D(\nu 00;\, \alpha\beta\gamma) = P_\nu(\cos\beta)$, where P_ν are the Legendre polynomials and β is the angle between the two radiations, becomes:

$$(5)\qquad W(\beta) \sim \sum (-1)^{L_1+L_2+\nu} c_{\nu 0}(L_1L_1')\, c_{\nu 0}(L_2L_2')(j_1\|L_1\|j)(j_1\|L_1'\|j)(j_2\|L_2\|j)(j_2\|L_2'\|j)\, \cdot$$
$$\cdot\, W(jjL_1L_1';\, \nu j_1)\, W(jjL_2L_2';\, \nu j_2)\, P_\nu(\cos\beta)\,.$$

Where the sum is taken over L_1, L_1', L_2, L_2', ν and where $c_{\nu 0}(L_1L_1')$ stands for ${}^{\mathrm{LC}}c_{\nu 0}(L_1L_1')$ or ${}^{\mathrm{RC}}c_{\nu 0}(L_1L_1')$ accordingly we observe the LC or the RC polarization of the first quantum γ_1.

The same thing happens for $c_{\nu 0}(L_2L_2')$.

Keeping in mind the expressions for ${}^{\mathrm{LC}}c_{\nu 0}$ and ${}^{\mathrm{RC}}c_{\nu 0}$, we come to the following conclusions (some of which immediately follow from the parity conservation):

a) the angular correlations with circular polarization measurement of the two quanta do not give informations on the parity of the nuclear levels;

b) the measurement of the LC polarization of a quantum and of the RC polarization of the other one lead to the same correlation function that we would obtain by inverting the two analyzers of circular polarization;

c) the measurement of LC polarization of the two quanta leads to the same correlation function obtained from the measurement of RC polarization of both quanta. One can verify immediately that such correlations remain unchanged after the exchange of γ_1 and γ_2. Generally (see also *b*)), we can therefore ignore the fact that either γ_1 or γ_2 have entered into the analyzer;

d) the correlation function for the case in which the LC polarization of a quantum and the RC polarization of the other one are measured (to be indicated with W(LC-RC) or W(RC-LC) accordingly if we observe the LC (RC) polarization of γ_1 (γ_2) or of γ_2 (γ_1)) differs from the one in which we measure the LC or RC polarization of both quanta (to be indicated respectively with W(LC-LC) or W(RC-RC)) for a change in sign of the terms with ν odd in the eq. (5);

e) the presence of the Legendre polynomials of odd order in the eq. (5) implies an asymmetry around the values $\beta=\pi/2$. From what stated in *b*), *c*), *d*) it follows that:

$$\frac{W(\text{LC-LC})_{\beta=(\pi/2)+\theta}}{W(\text{LC-LC})_{\beta=(\pi/2)-\theta}}=\frac{W(\text{RC-RC})_{\beta=(\pi/2)+\theta}}{W(\text{RC-RC})_{\beta=(\pi/2)-\theta}}=$$

$$=\frac{W(\text{LC-RC})_{\beta=(\pi/2)-\theta}}{W(\text{LC-RC})_{\beta=(\pi/2)+\theta}}=\frac{W(\text{RC-LC})_{\beta=(\pi/2)-\theta}}{W(\text{RC-LC})_{\beta=(\pi/2)+\theta}}$$

where θ is an angle between 0 and $\pi/2$.

For pure multipole radiations, the eq. (5) is simplified and becomes:

$$W(\beta)\sim\sum(-1)^{L_1+L_2+\nu}c_{\nu 0}(L_1L_1)c_{\nu 0}(L_2L_2)(j_1\|L_1\|j\|)^2(j_2\|L_2\|j)^2\cdot$$
$$\cdot W(jjL_1L_1;\nu j_1)W(jjL_2L_2;\nu j_2)P_\nu(\cos\beta)\,.$$

The reduced matrix elements $(j\|L_1\|j)$ and $(j_2\|L_2\|j)$ are now constant factors and can be omitted. Therefore we can write:

$$W(\text{LC-RC})=W(\text{RC-LC})\sim\sum_\nu A_\nu P_\nu(\cos\beta)\,,\tag{6a}$$

$$W(\text{LC-LC})=W(\text{RC-RC})\sim\sum_\nu(-1)^\nu A_\nu P_\nu(\cos\beta)\,,\tag{6b}$$

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with:

$$A_\nu = (-1)^{j_1+j_2-2j}(2j+1)(2L_1+1)(2L_2+1)\,C(L_1L_1\nu;\,1\;-1)\cdot$$
$$\cdot\,C(L_2L_2\nu;\,1\;-1)\,W(jjL_1L_1;\,\nu j_1)\,W(jjL_2L_2;\,\nu j_2)\,.$$

The normalization factor $(-1)^{j_1+j_2-2j}(2j+1)(2L_1+1)(2L_2+1)$ has been introduced in order to make $A_0 = 1$.

The expression of A_ν shows that it can be split into the product of a term depending only from the first transition and a term depending only from the second one. We can write in detail (BR, Sect. III-A):

$$A_\nu = F_\nu(L_1j_1j)\,F_\nu(L_2j_2j)\,,$$

where

$$F_\nu(L_1j_1j) = (-1)^{j_1-j-1}(2j+1)^{\frac{1}{2}}(2L_1+1)\,C(L_1L_1\nu;\,1\,-1)\,W(jjL_1L_1;\,\nu j_1)$$

and similarly for $F_\nu(L_2j_2j)$.

This splitting is also useful for the numerical calculation because it is more convenient to tabulate the coefficients F_ν rather than A_ν. A complete tabulation, but only relative to even values of ν, can be found in the above-mentioned work of BIEDENHARN and ROSE.

3 – Application to the γ-γ cascade of ^{60}Co.

The transition is of the type 4(2)2(2)0 so that $L_1 = L_2 = 2$, $j_1 = 4$, $j = 2$ and $j_2 = 0$. The values of ν which are involved are $\nu = 0, 1, 2, 3, 4$ (to be remembered that $0 \leqslant \nu \leqslant \mathrm{Min}\,(2j,\, 2|j_1 - j|,\, 2|j_2 - j|)$).

Therefore, in order to know the correlation function ($6a$) or ($6b$) it is necessary to determine; $F_\nu(242)$ and $F_\nu(202)$. Now we have:

$F_0(242) = 1$	$F_0(202) = 1$	$A_0 = 1$
$F_1(242) = 0.4714$	$F_1(202) = -0.7071$	$A_1 = -0.3333$
$F_2(242) = -0.1707$	$F_2(202) = -0.5976$	$A_2 = 0.1020$
$F_3(242) = -0.1010$	$F_3(202) = 1.4142$	$A_3 = -0.1428$
$F_4(242) = -0.0085$	$F_4(202) = -1.069$	$A_4 = 0.0091$

so that:

$$W(\mathrm{LC\text{-}RC}) = W(\mathrm{RC\text{-}LC}) = 1 - 0.3333\,P_1(\cos\beta) + 0.1020\,P_2(\cos\beta) +$$
$$-\,0.1428\,P_3(\cos\beta) + 0.0091\,P_4(\cos\beta)\,,$$

$$W(\text{LC-LC}) = W(\text{RC-RC}) = 1 + 0.3333\,P_1(\cos\beta) + 0.1020\,P_2(\cos\beta) + \\ + 0.1428\,P_3(\cos\beta) + 0.0091\,P_4(\cos\beta).$$

The asymmetry, for $\beta = 90° \pm 90°$ is:

$$\frac{W(\text{LC-RC})_{\beta=180°}}{W(\text{LC-RC})_{\beta=0°}} = \frac{W(\text{LC-LC})_{\beta=0°}}{W(\text{LC-LC})_{\beta=180°}} \sim 2.50\,,$$

and for $\beta = 90° \pm 45°$:

$$\frac{W(\text{LC-RC})_{\beta=135°}}{W(\text{LC-RC})_{\beta=45°}} = \frac{W(\text{LC-LC})_{\beta=45°}}{W(\text{LC-LC})_{\beta=135°}} \sim 1.52\,.$$

The strong asymmetries so obtained do not correspond, of course, to the actual experimental situation since they refer to the case of perfect analyzers: that is, analyzers for which we can say that, if a photon has been accepted, then most certainly it is in the pure state of circular polarization, as indicated by the analyzer.

The presence of the Legendre polynomials of odd order in the double correlation function is not limited to the case we have just treated. They also appear if, for example, one or both the γ-rays are replaced by electrons (particles of spin $\frac{1}{2}$) of which the longitudinal polarization state is observed. Naturally, beside the above-mentioned cases, the presence of the Legendre polynomials of odd order can be caused by parity non-conservation in the nuclear cascade.

* * *

I wish to express my deepest gratitude to Prof. A. Borsellino for the affectionate help he has given me in this work, as well as during my studies and preparation.

RIASSUNTO

Viene esaminato il problema della doppia correlazione angolare γ-γ in cui si osserva lo stato di polarizzazione circolare dei fotoni e si discute la relazione tra le varie possibili correlazioni. La presenza dei polinomi di Legendre di ordine dispari introduce un'asimmetria intorno al valore $\beta = \pi/2$, dove β è l'angolo fra le due radiazioni. Si sono fatti calcoli numerici per la cascata γ-γ del ^{60}Co.

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Dirac Matrices in Riemannian Space.

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Summary. — An explicit formula is developed for the affine spinor connection, Γ_r, in an arbitrary spin representation. This formula makes it possible explicitly to write the Dirac equation in Riemannian space, $\gamma^i(\Psi_{,i} - \Gamma_i\Psi) + m\Psi = 0$, in a form which may be subjected to arbitrary similarity transformations. Also, some useful algebraic properties of Dirac matrices are developed.

1. – Introduction.

The increased interest during the past few years in the relationship between general relativity and quantum mechanics has naturally produced a corresponding interest in the relationship between spinors and general relativity. The formalism of this relationship has been discussed adequately by several authors (1). In order to properly treat derivatives of spinors this formalism requires the introduction of an affine spinor connection, Γ_r, just as the proper treatment of derivatives of tensors requires the introduction of the usual affine connection (Christoffel symbols). It would be useful to have a general and explicit formula for Γ_r in terms of the Dirac matrices. The purpose of this paper is to develop such a formula (equation (30), or equations (37) and (39)).

If one limits oneself to a special class of spinor representations (the *n-bein* representations), the expression for Γ_r is quite simple and has been given in various forms by Bargmann (1), Klein (2), and others (see equation (40)).

(*) National Science Foundation Predoctoral Fellow.

(1) For a further discussion of the formalism used in the present article, see for example V. Bargmann: *Sitzber. preuss. Akad. Wiss., Phys.-math. Kl.* (Berlin, 1932), p. 346, or D. R. Brill and J. A. Wheeler: *Rev. Mod. Phys.*, **29**, 465 (1957).

(2) O. Klein: *Arch. Math. Astr. och Phys.*, **34**, 1 (1947).

To attempt to find Γ_r when the Dirac matrices are given in a general representation, by transforming to an *n-bein*, finding Γ_r in that system, and then transforming back to the original system, is possible. However, it is much more difficult and less elegant than using the formula given here.

In the course of this article several algebraic properties of Dirac matrices will be found, which may have further applications than are considered here. The formulae are developed for n dimensions so as to be of use, not only in the standard four-dimensional general relativity, but also in theories requiring more (or less) dimensions. The formalism used here reduces in the four-dimensional case to the so-called «four-component» (in contrast with «two-component») spinor theory.

Following is a brief review of some features of spinor formalism in Riemannian space which are relevant to the subject of this paper: Dirac matrices in Riemannian n-space are a vector field of matrices, γ_i, restricted only by the condition that the anticommutator of the vector with itself at a fixed point be a multiple of the unit matrix, γ:

$$\{\gamma_i, \gamma_j\} = 2g_{ij}\gamma \,. \tag{1}$$

The symmetric tensor g_{ij} is interpreted as the metric tensor of the space. The array of quantities g_{ij} is assumed to have a non-zero determinant so that g^{ij} may be defined in the usual fashion. The tensors g^{ij} and g_{ij} are used in the usual manner to raise and lower indices; also, they are used in the customary way to define Christoffel symbols, $\Gamma_{ij}^{\ \ k}$, and hence to define covariant differentiation. Partial differentiation will be symbolized with a comma (,) and covariant differentiation with a semi-colon (;). So, for example,

$$\gamma_{i;r} = \gamma_{i,r} - \Gamma_{ir}^{\ \ k}\gamma_k \,. \tag{2}$$

The following relation will be of use:

$$\{\gamma_{i,r}, \gamma_j\} + \{\gamma_i, \gamma_{j;r}\} = 2g_{ij;r} = 0 \,. \tag{3}$$

A spinor and its covariant derivative do not have the same behavior under a similarity transformation. Hence one introduces the total derivative, symbolized with a vertical stroke (|), which for a column spinor of any tensor character is defined as

$$\Psi_{|r} = \Psi_{;r} - \Gamma_r\Psi \,. \tag{4}$$

Under similarity transformation Ψ goes to $S\Psi$; in order that $\Psi_{|r}$ likewise go to $S\Psi_{|r}$, it is assumed that the matrix vector Γ_r go to $S\Gamma_r S^{-1} + S_{,r}S^{-1}$. It is the total derivative which appears in the Dirac equation. The distributive

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law is assumed to hold for the total derivative. Hence for a row spinor, Φ (which under similarity transformation goes to ΦS^{-1}), it must be that

$$\Phi_{|r} = \Phi_{;r} + \Phi\Gamma_r\,, \tag{5}$$

in order that the product $\Phi\Psi$, which is unchanged by a similarity transformation, have a total derivative equal to its covariant derivative. Since a matrix, M, should behave like the product $\Psi\Phi$, it follows that

$$M_{|r} = M_{;r} - [\Gamma_r, M]\,. \tag{6}$$

Just as Christoffel symbols are determined by the requirement that the covariant derivative of the metric tensor vanish, Γ_r is determined by the requirement (which is easily seen to be invariant under similarity transformation) that

$$\gamma_{i|r} = \gamma_{i;r} - [\Gamma_r, \gamma_i] = 0\,. \tag{7}$$

The remainder of this paper will be devoted to finding an explicit expression for Γ_r which satisfies (7).

By multiplying (7) from the right by γ^i, one finds

$$\xi\Gamma_r = \tfrac{1}{2}\gamma_{i;r}\gamma^i\,, \tag{8}$$

where ξ is an operator which acting on a matrix M is defined by

$$\xi M = \tfrac{1}{2}(nM - \gamma_i M\gamma^i)\,. \tag{9}$$

By n is always meant the dimension of the Riemannian space. The problem now reduces to finding the inverse of ξ. The next section concerns this question.

2. – Algebra of ξ.

First, define the following

$$\gamma(m)_{ijk\ldots} = (1/m!)A(ijk\ldots)\gamma_i\gamma_j\gamma_k\ldots \tag{10}$$

where there are m indices $ijk\ldots$. The operator A $(ijk\ldots)$ mean that the expression following it should be totally antisymmetrized in $ijk\ldots$; that is, the following expression should be replaced by the sum of all expressions obtained from it by even permutation of $ijk\ldots$ minus the sum of all expressions obtained from it by odd permutation of $ijk\ldots$. The tensor $\gamma(m)_{ijk\ldots}$ is totally antisym-

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metric and of rank m; hence it vanishes if m exceeds n; $\gamma(1)_i = \gamma_i$, $\gamma(0) = \gamma$. By going to a locally Cartesian co-ordinate frame and considering separately the cases when $a \dots bc \dots d$ and $i \dots jk \dots l$ have no indices in common, one index in common, two indices, etc., one may easily show

$$(11) \qquad \gamma(r)_{a\dots bc\dots d}\gamma(s)_{i\dots jk\dots l} = A(a \dots bc \dots d)\, A(i \dots jk \dots l) \cdot \\ \cdot \sum_m [1/m!\,(r-m)!\,(s-m)!] g_{di} \dots m \text{ factors} \dots g_{cj}\gamma(r+s-2m)_{a\dots bk\dots l} \cdot$$

For t less than sero, $(1/t!)$ is considered to vanish. The formula for the commutator of $\gamma(r)_{a\dots d}$ and $\gamma(s)_{i\dots l}$ may be obtained by doubling the right side of (11) and, if the product rs is odd, omitting the terms for odd m, or, if rs is even, omitting the terms for even m. A similar rule for obtaining the anti-commutator may be found by interchanging the phrases « odd m » and « even m » in the above rule.

Next define the r-th cross-sum on a matrix M as

$$(12) \qquad X_r M = (1/r!)\, \gamma(r)_{a\dots c} M \gamma(r)^{c\dots a} .$$

By rewriting $\gamma_a\gamma(r)_{i\dots k}$ by means of (11) in $X_1 X_r M = (1/r!)\gamma_a\gamma(r)_{i\dots k} M\gamma(r)^{k\dots i}\gamma^a$, one finds

$$(13) \qquad (r+1)X_{r+1} = (n-2\xi)X_r - (n-r+1)X_{r-1} ,$$

where X_1 has been rewritten in terms of ξ by means of (9). It is now asserted that, as a function of ξ,

$$(14) \qquad X_r = \Omega_{r\xi}(n) ,$$

where $\Omega_{r\xi}(n)$ is a polynomial of degree r in ξ defined by

$$(15) \qquad \omega(\xi, n, z) = (1-z)^\xi (1+z)^{n-\xi} = \sum_r \Omega_{r\xi}(n) z^r .$$

In this last expression, ξ is treated as a number. To prove this assertion, consider the equality,

$$(16) \qquad [(\mathrm{d}/\mathrm{d}z) - (n-2\xi) + nz - z^2(\mathrm{d}/\mathrm{d}z)]\, \omega(\xi, n, z) = 0 ,$$

which may be verified by direct computation. If the series expansion of (15) is substituted into this, one finds that $\Omega_{r\xi}(n)$ obeys the same two-step recursion relation in r (13) as does X_r. Thus if (14) can be shown to hold for two values of r it will be true for all r. It is easily verified that $X_0 = \Omega_{0\xi}(n) = 1$ and $X_1 = \Omega_{1\xi}(n) = n - 2\xi$.

Application of the binomial theorem to (15) shows

$$\Omega_{r\xi}(n) = \sum_m (-1)^m {}_\xi C_{m\,n-\xi} C_{r-m}\,, \tag{17}$$

where ${}_\xi C_m$ is the coefficient of z^m in the power series expansion of $(1+z)^\xi$ (ξ being treated as a number). Using the well-known expressions for ${}_\xi C_m$, one may calculate the case $r=n+1$ in (17) without too much difficulty to yield

$$X_{n+1} = \Omega_{(n+1)\xi}(n) = [2^{n+1}/(n+1)!](n-\xi)(n-1-\xi)\dots(1-\xi)(-\xi) = 0\,. \tag{18}$$

The expression has been set equal to zero since X_{n+1} is the zero operator, as can be seen from its definition and the fact that $\gamma(n+1)_{ijk\dots}$ is zero. Thus it is seen that not all powers of ξ are linearly independent. If (18) is multiplied through by an arbitrary polynomial in ξ, the result may be stated as follows: Any polynomial in ξ which, if ξ where a number, would vanish when ξ equalled any of the integers zero to n is the zero operator. From this it is but a simple step to the conclusion which is the main point of this section. Any two polynomials in the operator ξ are equal if they have equal numerical values when ξ is replaced by any of the integers zero to n.

The following notation will prove useful. By $\langle f(\xi)\rangle(n)$ is meant that polynomial of degree n in ξ which equals the function $f(\xi)$ when ξ is replaced by any integer from zero to n. For example, if $n=2$, $\langle 2^\xi\rangle(n) = \frac{1}{2}\xi^2+\frac{1}{2}\xi+1$. From the theorem in the preceding paragraph it follows that

$$\langle f(\xi)\rangle(n)\langle g(\xi)\rangle(n) \quad = \langle f(\xi)g(\xi)\rangle(n)\,, \tag{19}$$

$$\langle f(\xi)\rangle(n) + \langle g(\xi)\rangle(n) = \langle f(\xi) + g(\xi)\rangle(n)\,. \tag{20}$$

These last two results make calculations involving ξ quite easy. If $f(\xi)$ is a constant, then it is clear that $\langle f(\xi)\rangle(n) = f(\xi)$.

The following algebraic results will be needed later. Making use of (11) in a manner similar to that used in deriving (13), one may show that for any M

$$[\xi M, \gamma_i] = (n+1-\xi)[M, \gamma_i]\,, \tag{21}$$

$$\{\xi M, \gamma_i\} = (n-1-\xi)\{M, \gamma_i\}\,. \tag{22}$$

If one takes a polynomial of degree $n+1$ in ξ which equals zero (such as appears in (18)), applies it to M, and then commutes or anticommutes the result with γ_i, one finds with the use of (21) and (22) that

$$\langle \delta_{0\xi}\rangle(n)[M, \gamma_i] = 0\,, \qquad \langle \delta_{n\xi}\rangle(n)\{M, \gamma_i\} = 0\,. \tag{23}$$

The new symbol in (23) is the well-known Kronecker delta. One may now generalize (21) and (22) to

$$[\langle f(\xi)\rangle(n)M, \gamma_i] = \langle f(n+1-\xi)\rangle(n)\,[M, \gamma_i]\,, \tag{24}$$

$$\{\langle f(\xi)\rangle(n)M, \gamma_i\} = \langle f(n-1-\xi)\rangle(n)\,\{M, \gamma_i\}\,. \tag{25}$$

The proof of these is quite straightforward, but there is a small detail which should not be overlooked. For instance, starting with (21), one can immediately obtain an expression like (24) in which the function appearing on the right equals $f(n+1-\xi)$ when ξ is replaced by the integers *one* to $n+1$. One must then use the freedom offered by the first of equations (23) to get (24).

The following results will not be needed later, but are interesting. With the use of the rules for commutators and anticommutators following (11), one may show that

$$\xi\gamma(s)_{ijk\ldots} = s\gamma(s)_{ijk\ldots} \qquad \text{if } s \text{ is even,} \tag{26}$$

$$= (n-s)\gamma(s)_{ijk\ldots} \qquad \text{if } s \text{ is odd.} \tag{27}$$

Also, since the factors in a product may be cyclically permuted before the trace of the product is taken, it follows that

$$\mathrm{Tr}\,(\xi M) = 0\,. \tag{28}$$

3. – Solution for Γ_r.

It would appear from (19) that Γ_r could be obtained by acting on both sides of (8) with $\langle 1/\xi\rangle(n)$. Unfortunately, $\langle 1/\xi\rangle(n)$ is not defined, since $(1/0)$ has no meaning. So consider instead the « next best thing », $\langle 1/\xi\rangle_a(n)$, which by definition equals $\langle a/(a\xi+\delta_{0\xi})\rangle(n)$; that is, it is that polynomial of degree n in ξ which equals $1/\xi$ when ξ is replaced by any integer from one to n and equals the given arbitrary number a when ξ is replaced by zero. Applying this to both sides of (8), one finds

$$\Gamma_r - \langle\delta_{0\xi}\rangle(n)\Gamma_r = \tfrac{1}{2}\langle 1/\xi\rangle_a(n)\gamma_{i;r}\gamma^i. \tag{29}$$

This equation does not determine $\langle\delta_{0\xi}\rangle(n)\Gamma_r$, so set it equal to $\langle\delta_{0\xi}\rangle(n)A_r$, where A_r is an arbitrary matrix:

$$\Gamma_r = \tfrac{1}{2}\langle 1/\xi\rangle_a(n)\gamma_{i;r}\gamma^i + \langle\delta_{0\xi}\rangle(n)A_r\,. \tag{30}$$

All that has been shown up till now is that if Γ_r exists it must be of the form (30). It will now be verified that (30) is indeed a solution of (7). Commuting both sides of (30) with γ_j and using (24), one finds

$$[\Gamma_r, \gamma_j] = \tfrac{1}{2}\langle 1/(n+1-\xi)\rangle(n)[\gamma_{i;r}\gamma^i, \gamma_j] \,. \tag{31}$$

But if one multiplies (3) from the right by γ^i, one finds

$$[\gamma_{i;r}\gamma^i, \gamma_j] = 2(n+1-\xi)\gamma_{j;r} \,. \tag{32}$$

This together with (31) verifies (30).

The remaining remarks are intended as aids in the calculation of Γ_r. It may be of use to reexpress $\langle 1/\xi\rangle_a(n)$ by application of the following general formula:

$$\langle f(\xi)\rangle(n) = \sum_r f(r)\langle\delta_{r\xi}\rangle(n) \,. \tag{33}$$

In conjunction with (33) one might employ

$$\langle\delta_{r\xi}\rangle(n) = (\tfrac{1}{2})^n \sum_m \Omega_{rm} X_m \,. \tag{34}$$

This can be verified by making the substitution $z = (1-y)/(1+y)$ in (15) which, if ξ is an integer from zero to n, gives a valid power series in y:

$$(\tfrac{1}{2})^n \sum_r \Omega_{r\xi}(n)\omega(r, n, y) = y^\xi. \tag{35}$$

Substituting (15) into (35) one obtains an expression in $\Omega_{r\xi}$ which, using (14), reduces to (34).

The following useful recursion formula follows immediately from the definition of $\langle 1/\xi\rangle_a(n)$:

$$\langle 1/\xi\rangle_a(n) = (1/n)[1 + (n-\xi)\langle 1/\xi\rangle_b(n-1)] \,, \tag{36}$$

where $a = b + (1/n)$. Probably the easiest expression from which to calculate Γ_r is

$$\Gamma_r = F_n(X_1)\gamma_{i;r}\gamma^i \,. \tag{37}$$

For simplicity A_r has been chosen to be zero. From (30) and (36) follows [3]

$$F_n(X_1) = (1/2n)[1 + (X_1 + n)F_{n-1}(X_1 - 1)] \,. \tag{38}$$

---

[3] This expression was first obtained by V. Bargmann.

It is most convenient to choose $F_0(X_1)$, which is arbitrary, to be zero. For small n, one obtains

$$(39)\qquad \begin{cases} F_0(X_1) = 0, \quad F_1(X_1) = \frac{1}{2}, \quad F_2(X_1) = (1/8)(X_1 + 4), \\ F_3(X_1) = (1/48)(X_1^2 + 6X_1 + 17), \\ F_4(X_1) = (1/384)(X_1^3 + 8X_1^2 + 28X_1 + 96), \\ F_5(X_1) = (1/3840)(X_1^4 + 10X_1^3 + 40X_1^2 + 150X_1 + 759). \end{cases}$$

If special assumptions are made, the formula for Γ_r can often be simplified. For instance, *n-bein* formalism in effect assumes that $\langle\delta_{r\xi}\rangle(n)\gamma_{i;r}\gamma^i = \delta_{2r}\gamma_{i;r}\gamma^i$. In this case, using (33), one may write (30) as

$$(40)\qquad \Gamma_r = \tfrac{1}{4}\gamma_{i;r}\gamma^i.$$

The assumption that an *irreducible* representation of the γ_i matrices is being employed has important consequences *only if n is odd*, in which case it implies that the operators X_r and X_{n-r} are identical for all r. This can be shown to imply that two polynomials in ξ are equal if they are equal when ξ is replaced by any *even* integer zero to $n-1$. Obvious changes may now be made in the definition of $\langle f(\xi)\rangle(n)$ and in equations (36) and (38). In conjunction with (37) one may now use, instead of (39),

$$(41)\qquad F_1(X_1) = 0, \qquad F_3(X_1) = \tfrac{1}{4}, \qquad F_5(X_1) = (1/32)(X_1 + 7).$$

* * *

The author wishes to thank JOHN R. KLAUDER and Dr. V. BARGMANN for many helpful discussions.

RIASSUNTO (*)

Si sviluppa una formula esplicita per la connessione spinoriale affine, Γ_r, in una rappresentazione spinoriale arbitraria. Questa formula permette di scrivere l'equazione di Dirac nello spazio riemanniano, $\gamma^i(\Psi_{,i} - \Gamma_i\Psi) + m\Psi = 0$, in una forma cui possono imporsi trasformazioni di similitudine arbitrarie. Si sviluppano anche alcune utili proprietà algebriche delle matrici di Dirac.

(*) *Traduzione a cura della Redazione.*

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Decay of Hyper-^{5}He.

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Summary. — Energy and angular distributions of the particles arising from a number of cases of the decay of $^5\mathrm{He}_\Lambda$ have been analyzed. The influence of the strong spin-orbit coupled ^{4}He-^{1}H state on the decay of $^5\mathrm{He}_\Lambda$ is noted.

All of the known cases (1-5) of « uniquely defined » (6) decays of $^5\mathrm{He}_\Lambda$ are quite similar, not only in the energies but also in the relative orientations of the product particles. For instance, it is observed in the nine events studied here, that the mean energies of the ^{4}He, ^{1}H and π^- particles produced in the decay of $^5\mathrm{He}_\Lambda$ are: (3.55 ± 1.15) MeV, (6.34 ± 1.91) MeV and (24.5 ± 3.3) MeV, respectively, and the mean angles between the ^{4}He and π^- and between the ^{4}He and ^{1}H particles are: $150° \pm 12.5°$ and $145° \pm 11°$, respectively. (The varia-

(*) Assisted by a joint program of the U. S. Atomic Energy Commission and the U. S. Office of Naval Research.

(1) R. D. Hill, E. O. Salant, M. Widgoff, L. S. Osborne, A. Pevsner, D. M. Ritson, J. Crussard and W. D. Walker: *Phys. Rev.*, **94**, 797 (1954); **101**, 1127 (1956).

(2) J. Crussard, V. Fouché, G. Kayas, L. Leprince-Ringuet, D. Morellet, F. Renard and J. Trembley: *Suppl. Nuovo Cimento*, 4, 616 (1956); *Proc. Rochester Conference*, ch. V (1956), p. 39.

(3) P. H. Fowler and D. H. Perkins: *Suppl. Nuovo Cimento*, **4**, 487 (1956); *Nuovo Cimento*, **4**, 158 (1956).

(4) W. F. Fry, J. Schneps and M. S. Swami: *Phys. Rev.*, **101**, 1526 (1956).

(5) W. Slater, E. Silverstein, R. Levi-Setti and V. L. Telegdi: paper presented Nov. 23, 1956 at the *Chicago Meeting of the American Physical Society*.

(6) A. Filipkowski, J. Gierula and P. Zielinski: *Acta Phys. Pol.*, **16**, 139 (1957).

tions given are mean deviations from the means. They represent observed variations which are, in fact, considerably larger than the experimental errors involved in the observations). From an assumed value of $Q = 36.9$ MeV, for the free decay of the Λ^0, the mean binding energy of the Λ^0 in ${}^5\mathrm{He}_\Lambda$ is found to be (2.5 ± 0.4) MeV.

A reasonable approach towards analyzing the decay of ${}^5\mathrm{He}_\Lambda$ would appear to be to regard the hyperfragment as a loosely knit structure of α-particle and Λ^0. Since the Λ^0 is not excluded by Pauli's principle from occupying the lowest s state, it would seem to be that the level at -2.5 MeV is a $1s$ state. Decay of ${}^5\mathrm{He}_\Lambda$ probably occurs with a lifetime which is not significantly different from that of the free Λ^0 and, since the binding energy of the Λ^0 is so small compared with the Q of the Λ^0, one might expect that to a first order the decay of ${}^5\mathrm{He}_\Lambda$ would be merely the independent decay of the Λ^0 in flight within the hyperfragment. However, when one transforms to a system in which the Λ^0 is at rest, using as a velocity that which is required to conserve momentum of the Λ^0 with the recoiling α-particle, one finds that there is very strong evidence that the Λ^0 decay does not take place independently of the residual ${}^4\mathrm{He}$ nucleus. In the rest system of the assumed recoiling Λ^0 particle one finds that the proton is projected with an average momentum of (66.5 ± 11) MeV/c at an angle of $46.5° \pm 10°$ to the recoil direction of the α-particle and no case has been found in which the angle is greater than 65°

Another argument against the completely independent decay of the Λ^0 appears to be the particular form of the ${}^4\mathrm{He}$ recoil spectrum, which is shown for the nine events in Fig. 1. If the ${}^4\mathrm{He}$ and Λ^0 particles were in a 2.5 MeV bound $1s$ state, we have found for a number of possible potential wells of various shapes and widths that the momentum distribution would favor the region below 100 MeV/c. The average ${}^4\mathrm{He}$ recoil momentum is observed to be (160 ± 25) MeV/c. It is unlikely that any experimental difficulty of observing low momentum α-particle recoils in nuclear emulsions could explain the paucity of events in the region below 120 MeV/c, which is the lowest observed recoil momentum. Recoils of approximately 2 to 3 μm, corresponding to α-particle momenta of 60 to 80 MeV/c, should certainly be observable, yet the shortest recoil track observed was 6 μm.

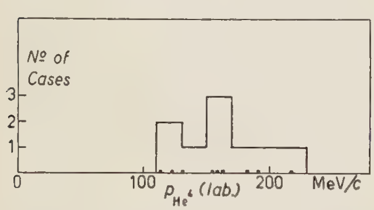

Fig. 1. – ${}^4\mathrm{He}$ recoil momentum spectrum from nine ${}^5\mathrm{He}_\Lambda$ decay events.

In order to investigate the influence of the ${}^4\mathrm{He}$ particle on the emitted proton which is presumably projected from the Λ^0, we transform into the center of mass of the ${}^4\mathrm{He}$-${}^1\mathrm{H}$ system. The sums of the kinetic energies of the α-particle and proton, in their own center of mass system, for the nine events

are shown in Fig. 2. It is immediately apparent that the average energy of the ^{4}He-^{1}H system corresponds to the wellknown $p\frac{1}{2}$ resonance of these two particles. According to the phase shift analysis of DODDER and GAMMEL (7), the ^{4}He-^{1}H system has a $p\frac{3}{2}$ state at 2.1 MeV (c.m.) and a $p\frac{1}{2}$ state at 9.6 MeV (c.m.). The $p\frac{1}{2}$ state, moreover, is very wide; according to AJZENBERG and LAURITSEN (8), of the order of 5 MeV (lab.). The observed average energy of the ^{4}He and ^{1}H particles together is 9.1 MeV (c.m.) and the mean variation from the mean is 3.1 MeV (c.m.). It appears, therefore, that the proton from the Λ^0 decay is attracted towards the ^{4}He particle by a nuclear force modified by the strong spin-orbit coupling between the ^{4}He and ^{1}H particles.

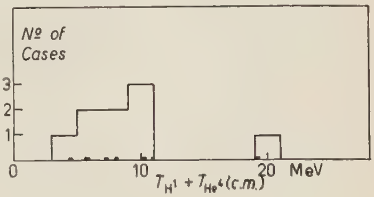

Fig. 2. – Distribution of sum of ^{4}He and ^{1}H kinetic energies from nine ^{5}He$_\Lambda$ decay events

From the center of mass of the ^{4}He-^{1}H system, the emitted pion will always appear to recede in the opposite direction to the recoil of the ^{4}He-^{1}H system. An « apparent » scattering angle of the proton can thus be measured from the direction of recoil of the ^{4}He-^{1}H system to the direction of the outgoing « scattered » proton in the center of mass of the ^{4}He-^{1}H system. After correcting for the solid angle factor, the distribution of proton scattering angle in the center of mass of the ^{4}He-^{1}H system for the nine events studied is shown in Fig. 3. The distribution shows a strong backward scattering asymmetry which probably indicates an interference between s and p wave scattering and probably can be understood in terms of the negative s and positive p wave phase shifts already computed by DODDER and GAMMEL (7).

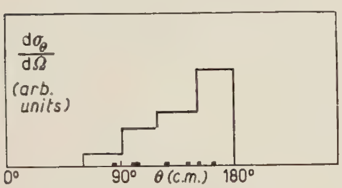

Fig. 3. - Angular distribution of number of ^{1}H particles scattered per unit solid angle. Angle θ is between the direction of emission of ^{1}H in the center of mass of the ^{4}He-^{1}H. system and the direction opposite to that of emission of the π from the decay of the ^{5}He$_\Lambda$.

It can readily be shown that the energy and angular characteristics of the ^{5}He$_\Lambda$ decay are determined by the following dynamical conditions: *a*) the ^{4}He-^{1}H system is produced in a spin-orbit coupled state, largely the $p\frac{1}{2}$ state of considerable width, *b*) the ^{4}He-^{1}H system collectively conserves momentum with the pion emitted from the ^{5}He$_\Lambda$ decay, *c*) the pion kinetic energy equals the residual energy from the Λ^0 free decay after providing for the binding

(7) D. C. DODDER and J. L. GAMMEL: *Phys. Rev.*, **88**, 520 (1952).
(8) F. AJZENBERG and T. LAURITSEN: *Rev. Mod. Phys.*, **27**, 77 (1955).

energy of the $^{5}\mathrm{He}_{\Lambda}$ and the energy of the $^{4}\mathrm{He}$-$^{1}\mathrm{H}$ state, *d*) the $^{4}\mathrm{He}$ and $^{1}\mathrm{H}$ particles conserve momentum in their own center of mass system after the scattering process and *e*) the angular distribution of the proton scattering by the $^{4}\mathrm{He}$ particle is a definite function of the scattering angle probably determined by the s and p wave phase shifts appropriate to the energies of the particles in the $^{4}\mathrm{He}$-$^{1}\mathrm{H}$ state.

RIASSUNTO (*)

Abbiamo analizzato le distribuzioni energetica e angolare delle particelle originate in alcuni casi di decadimento del $^{5}\mathrm{He}_{\Lambda}$. Si nota l'influenza dello stato $^{4}\mathrm{He}$-$^{1}\mathrm{H}$ con accoppiamento di spin-orbita forte sul decadimento del $^{5}\mathrm{He}_{\Lambda}$.

(*) *Traduzione a cura della Redazione.*

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Estimate of Λ-Nucleon Potential with Hard Core from the Binding Energy of Hypertriton.

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Summary. — A variational calculation of the binding energy of hypertriton is made in order to determine the strength of a Λ-nucleon potential with an infinite repulsive core. A hard core potential is also used to describe the proton-neutron interaction. A Λ-nucleon potential is found which is stronger than the Λ-nucleon potential determined from previous calculations which assumed monotonically varying potentials. Nevertheless, the Λ-nucleon potential still does not appear to be strong enough to bind the hyperdeuteron. Tensor forces are omitted in the calculation, but an argument is given to show that their omission should not substantially affect the results.

1. – Introduction.

Information about the Λ-nucleon (abbreviated ΛN) potential can be deduced from the experimentally measured binding energies of the Λ in hyperfragments. Dalitz (1) has given an analysis which assumes that the ΛN force:

a) has zero range;

b) can be represented by a central, spin-dependent potential, and

c) is sufficiently weak so that the nucleus in which it is bound is undistorted by its presence.

The method of Dalitz yields one parameter to describe the ΛN potential in each of the triplet and singlet spin states. (The spin of the Λ is taken to

(1) R. H. Dalitz: *Proceedings Sixth Annual Rochester Conference on High Energy Physics* (1956).

be $\frac{1}{2}$.) The parameter giving the strength of the ΛN potential can be taken to be the volume integral of the potential Ω, or the well depth parameter s. This latter parameter as defined by BLATT and JACKSON (2) is the number by which a potential must be divided in order to give a just-bound state of the two-body system. For $s > 1$, the system is bound; for $s < 1$, the system is unbound. Since in this paper we use a ΛN potential with an infinite repulsive core, Ω does not exist. Therefore we calculate s as a measure of the strength of the ΛN potential.

A number of other authors (3-7) have made analyses which are essentially similar to that of DALITZ, with variations such as the relaxing of assumption a). In this case a whole spectrum of values of Ω can be obtained as a function of the assumed range for the Λ-nucleon potential, Ω increasing with increasing range. The results, however, are not very sensitive to this range, as long as it is assumed to be small.

DALITZ, in his original work, recognized that special methods must be used to treat the hyperfragment $^3H_\Lambda$. This is because the underlying nucleus in this case, the deuteron, is a weakly-bound extended structure which can be distorted by the presence of the Λ. Various authors have calculated the binding energy of $^3H_\Lambda$ by perturbation (8) or variational (1,5,9,10) methods, relaxing assumption c). Assumption a) must also be relaxed in the variational calculations, since otherwise the calculated binding energy of $^3H_\Lambda$ will be infinite (11). The fact that some authors obtain finite binding energies with a zero ranged ΛN potential is due to the poorness of their trial wave functions.

The variational calculations have been of two types, classified according to the choice of a trial wave function Ψ for $^3H_\Lambda$. In the first type, Ψ is taken to be a product of a wave function Φ_d describing the relative position of the proton and neutron and a wave function Φ_Λ describing the relative position of the Λ and the center of mass of the deuteron:

$$\Psi = \Phi_d(\boldsymbol{r}_p - \boldsymbol{r}_n)\,\Phi_\Lambda(\boldsymbol{r}_\Lambda - \tfrac{1}{2}(\boldsymbol{r}_p + \boldsymbol{r}_n)) \,, \tag{1}$$

where $\boldsymbol{r}_p$, $\boldsymbol{r}_n$, and $\boldsymbol{r}_\Lambda$ are the positions of the proton, neutron and Λ respect-

(2) J. M. BLATT and J. D. JACKSON: *Phys. Rev.*, **76**, 18 (1949).
(3) G. H. DERRICK: *Nuovo Cimento*, **4**, 565 (1956).
(4) J. T. JONES and J. M. KELLER: *Nuovo Cimento*, **4**, 1329 (1956).
(5) L. BROWN and M. PESHKIN: *Phys. Rev.*, **107**, 272 (1957).
(6) Y. POC KWON, J. ÔBA and T. GOTÔ: *Nuovo Cimento*, **6**, 832 (1957).
(7) M. ROSS and D. B. LICHTENBERG: *Phys. Rev.* (to be published).
(8) M. ROSS and D. B. LICHTENBERG: *Proceedings, Second Midwest Conference on Theoretical Physics* (Iowa, 1957).
(9) B. W. DOWNS: *Bull. Amer. Phys. Soc.*, **2**, 175 (1957).
(10) R. H. DALITZ and B. W. DOWNS: *Bull. Amer. Phys. Soc.*, **3**, 20 (1958).
(11) L. H. THOMAS: *Phys. Rev.*, **47**, 903 (1935.

ively. An important disadvantage of this trial wave function is that it does not take into account correlations in the positions of the three particles. That is, the wave function Φ_Λ is independent of whether the two nucleons are near or far from the Λ, provided their center of mass is a given distance away.

The second type of variational calculation is with a trial wave function of the form

$$\Psi = \Psi(r_1, r_2, r_3)\,, \tag{2}$$

where r_1, r_2, and r_3 are the distances between the Λ and proton, Λ and neutron, and proton and neutron respectively. For simplicity, Ψ is sometimes assumed to be a simple product wave function of the form

$$\Psi = \Phi(r_1)\,\Phi(r_2)\,\Phi'(r_3)\,.$$

A wave function of type (2) has the advantage of treating the three particles on an equal footing, and of taking into account correlations between the particles.

However, a difficulty still remains concerning the assumed form of the ΛN and nucleon-nucleon (NN) potentials. What the difficulty is can best be seen by considering the comparable problem in ordinary ^{3}H.

Many authors have made variational calculations of the binding energy of ^{3}H, assuming NN potentials determined from the binding energy of the deuteron and from low energy NN scattering data. The low energy two-nucleon information, as is well known, enables one to determine the depth and ranges, but not the shape, of the NN potential in the singlet and triplet spin states. If simple monotonically decreasing (in absolute magnitude) central potentials which fit the NN data are assumed, the calculated binding energy of ^{3}H comes out larger than the experimental value ([12]). If the singularity of the NN potential is made greater (for example a Yukawa is chosen instead of a Gaussian), the binding energy increases further. Since a variational calculation gives a lower limit to the binding energy, the conclusion is that the form of the assumed NN potential must be wrong. (We neglect such complications as velocity dependent potentials and three-body forces.)

Subsequently, PEASE and FESHBACH ([13]) showed that if tensor forces are included in the NN potential, the calculated binding energy of ^{3}H can be reduced sufficiently to give agreement with experiment. However their result showed that the tensor force must be longer ranged than the central force: a short range tensor force gives too little binding. Alternatively, OHMURA,

([12]) N. SVARTHOLM: *Thesis* (Lund, 1945).
([13]) R. L. PEASE and H. FESHBACH: *Phys. Rev.*, **88**, 945 (1952).

MORITA, and YAMADA ([14,15]) showed that if hard cores are assumed in the NN potential, agreement can be obtained omitting tensor forces. The potential outside the hard core must not be too singular, for the very singular Levy central potential (adjusted to fit the NN data) gives too large a binding even with a hard core ([16]).

We now return to consideration of ${}^3\mathrm{H}_\Lambda$, being guided by the results found for ordinary ${}^3\mathrm{H}$. Unfortunately, the two body ΛN scattering data are at present insufficient to determine the depth and range of the ΛN potential. The procedure in variational calculations has therefore been to take the binding energy of ${}^3\mathrm{H}_\Lambda$ as given from experiments, and to determine one parameter of the ΛN potential. The assumed form of the NN potential has been similar to the NN potentials which gave too large a binding in ${}^3\mathrm{H}$. Therefore, unless the actual ΛN potential is a monotonically varying central potential, the parameter deduced from the ${}^3\mathrm{H}_\Lambda$ calculation will not necessarily be appropriate to describe the two-body ΛN system.

In this work, we make an estimate of the strength of the ΛN potential, assuming that both the ΛN and NN potentials have hard cores. There is at present no experimental necessity to introduce hard cores into the ΛN potential. Our motivation is based on meson theory, which says that if the NN potential has a hard core, the ΛN potential may also have one ([17]). Our method is to make a variational calculation using a simple trial wave function of type (2). A description of the calculation and of the results is given in Sect. **2**. Tensor forces are omitted in the potentials. A discussion of the results and a partial justification for the omission of tensor forces are given in Sect. **3**.

2. – Calculation of the ΛN potential.

The method of calculation of the ΛN potential follows very closely the work of OHMURA, MORITA and YAMADA ([14,15]) on ordinary ${}^3\mathrm{H}$. The Schrödinger equation for ${}^3\mathrm{H}_\Lambda$ is of the form

$$(T+V+B)\Psi = 0\,, \tag{3}$$

where T is the kinetic energy operator (the same as the K in reference ([14]) except for the difference due to the heavier mass of the Λ), V is the potential and $B = 3$ MeV is the binding energy. The result is not very sensitive to

([14]) T. KIKUTA (OHMURA), M. MORITA and M. YAMADA: *Progr. Theor. Phys.*, **15**, 222 (1956).

([15]) T. OHMURA, M. MORITA and M. YAMADA: *Progr. Theor. Phys.*, **17**, 326 (1957).

([16]) H. FESHBACH and S. I. RUBINOW: *Phys. Rev.*, **98**, 188 (1955).

([17]) D. B. LICHTENBERG and M. ROSS: *Phys. Rev.*, **107**, 1714 (1957).

the choice of B, since B is a small difference between two large numbers. The potential is of the form

$$V = V_\Lambda(r_1) + V_\Lambda(r_2) + V_t(r_3)\,, \tag{4}$$

where V_t is the NN triplet potential and V_Λ is the spin-averaged ΛN potential. Letting the ΛN potential in the triplet and singlet spin states be $V_{\Lambda t}$ and $V_{\Lambda s}$ respectively, V_Λ is given by

$$\begin{cases} V_\Lambda = V_{\Lambda t} & \text{if } \; V_{\Lambda t} > V_{\Lambda s} \;\; \text{(in absolute magnitude)} \\ V_\Lambda = \frac{1}{4} V_{\Lambda t} + \frac{3}{4} V_{\Lambda s} & \text{if } \; V_{\Lambda t} < V_{\Lambda s} \;\; \text{(in absolute magnitude)}\,. \end{cases} \tag{5}$$

We choose the potentials V_Λ and V_t to have the same functional form as in reference ([14]):

$$\begin{cases} V_\Lambda(r) = -V_{0\Lambda} \exp[-\alpha_\Lambda(r - D)]\,, & r > D, \\ V_\Lambda(r) = \infty\,, & r < D, \end{cases} \tag{6}$$

with $V_{0\Lambda}$, α_Λ and D constant. We choose V_t to be given by the same expression as V_Λ with $V_{0\Lambda}$ replaced by V_{0t} and α_Λ replaced by α_t. The radius of the repulsive core D is taken the same for V_Λ and V_t. The form for the potential is that of decreasing exponential outside the hard core. OHMURA *et al.* ([15]) have shown that if a Yukawa is taken instead, the results are not changed significantly, provided the core radius is not too small ($D \gtrsim 0.2 \cdot 10^{-13}$ cm). The trial wave function Ψ is chosen to be a simple product wave function of the type given in equation (2):

$$\begin{cases} \Psi = \prod_{i=1}^{3} \exp[-\mu(r_i - D)] - \exp[-\nu(r_i - D)]\,, & \text{all } r_i > D, \\ \Psi = 0\,, & \text{otherwise}, \end{cases} \tag{7}$$

where $\prod$ indicates a product and μ and ν are parameters to be varied. An obvious improvement in the wave function (7) could be made by replacing μ and ν by μ' and ν' for $i = 3$. However, we do not do this because of the added complexity of the calculation when the number of parameters is doubled.

The parameters of the potentials are taken from reference ([14]). We choose two sets:

$$\begin{cases} D = 0.2 \cdot 10^{-13}\ \text{cm}\,, & V_{0t} = 286\ \text{MeV}\,, \\ \alpha_t = 1.89 \cdot 10^{13}\ \text{cm}^{-1}\,, & \alpha_\Lambda = 1.90 \cdot 10^{13}\ \text{cm}^{-1}\,, \end{cases} \tag{8a}$$

$$\begin{cases} D = 0.4 \cdot 10^{-13}\ \text{cm}\,, & V_{0t} = 475\ \text{MeV}\,, \\ \alpha_t = 2.52 \cdot 10^{13}\ \text{cm}^{-1}\,, & \alpha_\Lambda = 2.40 \cdot 10^{13}\ \text{cm}^{-1}\,. \end{cases} \tag{8b}$$

This leaves $V_{0\Lambda}$ as the parameter of the ΛN potential to be determined. From $V_{0\Lambda}$ we can compute the well depth parameter s. The range of the tail of the ΛN potential α_Λ is longer than that assumed in most calculations. We choose the parameters of (8) so that we can make use of the table of results already given by Ohmura *et al.* (reference [14], Table VI) for the potential energies. We can also use Table VI for the kinetic energy, reducing the values of the table by the factor

$$F = (M/3)(2/M + 1/M_\Lambda)\,,$$

where M and M_Λ are the masses of the nucleon and Λ. This factor is simply the ratio of the reduced mass of ^{3}H to the reduced mass of $^3\mathrm{H}_\Lambda$.

The calculation yields the following ΛN potentials:

$$V_{0\Lambda} = 160\ \text{MeV}\,, \qquad D = 0.2\,, \tag{9a}$$

$$V_{0\Lambda} = 280\ \text{MeV}\,, \qquad D = 0.4\,. \tag{9b}$$

For comparison, we also give the NN singlet potential depth V_{0s} with the same core radii D and ranges α_Λ:

$$V_{0s} = 200\ \text{MeV}\,, \qquad V_\Lambda/V_s = 0.8\,, \tag{a}$$

$$V_{0s} = 330\ \text{MeV}\,, \qquad V_\Lambda/V_s = 0.85\,. \tag{b}$$

3. – Discussion.

With the ΛN potentials found in the previous section, we obtain the following values for the well depth parameter s:

$$s = 0.82 \qquad \text{if} \qquad D = 0.2\,, \tag{10a}$$

$$s = 0.89 \qquad \text{if} \qquad D = 0.4\,. \tag{10b}$$

From equations (5) and (10) we see that if $V_{\Lambda t} > V_{\Lambda s}$, a bound state of $^2\mathrm{H}_\Lambda$ does not exist. If $V_{\Lambda s} > V_{\Lambda t}$, we see that a bound state of $^2\mathrm{H}_\Lambda$ exists if the following inequality is satisfied:

$$V_{\Lambda t}/V_{\Lambda s} < 0.28 \qquad \text{if} \qquad D = 0.2\,, \tag{11a}$$

or

$$V_{\Lambda t}/V_{\Lambda s} < 0.56 \qquad \text{if} \qquad D = 0.4\,. \tag{11b}$$

However, OHMURA *et al.* estimate that their trial wave function gives too low a binding energy of ^{3}H by ~ 1 MeV. If this is also the case with our trial wave function (7), which is of the same form, then the ΛN potentials should be reduced about 5%. Making this reduction, we obtain as the condition for a bound state of $^2\mathrm{H}_\Lambda$:

$$s \approx 0.78 \,, \qquad V_{\Lambda t}/V_{\Lambda s} \lesssim 0.1 \quad \text{if} \quad D = 0.2 \,, \tag{12a}$$

$$s \approx 0.85 \,, \qquad V_{\Lambda t}/V_{\Lambda s} \lesssim 0.4 \quad \text{if} \quad D = 0.4 \,. \tag{12b}$$

In the previous analysis of hyperfragments (7), it was found that if $V_{\Lambda s} > V_{\Lambda t}$, the spin dependence of the ΛN force is about

$$V_{\Lambda t}/V_{\Lambda s} \approx 0.4 \,. \tag{13}$$

Comparing (12) with (13), we see that for a core radius $D = 0.2$, $^2\mathrm{H}_\Lambda$ is not bound, and with $D = 0.4$ $^2\mathrm{H}_\Lambda$ is on the borderline of being bound. It should be pointed out that the differences between (11*a*) and (11*b*) are due, not only to the difference in the assumed core radii for the ΛN potential, but to the difference in the core radius of the NN potential. For comparison with the values (11) and (12), we quote the value of s obtained in reference [7] assuming Gaussian potentials without a core: $s = 0.6$.

We now briefly discuss the omission of tensor forces. First we consider the omission of the NN tensor potential. In spite of the fact that this potential is known to be large, we do not believe we have made a serious error in using only a central potential. This is because the hard core central NN potential used has been adjusted not only to fit the two-body NN data *but* (approximately) *the binding energy of* ^{3}H *as well.* Thus this potential ought to be suitable (as far as binding energy calculations are concerned) even when the deuteron is distorted provided the distortion is not greater than in ^{3}H. This is the case in $^3\mathrm{H}_\Lambda$. We next consider the omission of tensor forces in ΛN potential. In this case we have no empirical knowledge about whether a tensor force exists. The assumed ΛN potential is therefore consistent with experiment. However, we shall make a few comments about possible ΛN tensor forces. If $V_{\Lambda s} > V_{\Lambda t}$, the effective ΛN potential in $^3\mathrm{H}_\Lambda$ is $V_\Lambda = \frac{3}{4}V_{\Lambda s} + \frac{1}{4}V_{\Lambda t}$. Then, since tensor forces exist only in the triplet state, even a relatively large tensor potential will not appreciably alter V_Λ. Meson theory indicates that $V_{\Lambda s}$ is in fact greater than $V_{\Lambda t}$ (*). Secondly, as mentioned previously, PEASE

(*) *Note added in proof.* A recent work by R. DALITZ and B. DOWNS (to be published) indicates that experimentally $V_{\Lambda s}$ is greater than $V_{\Lambda t}$, in agreement with theory.

and FEHSBACH have shown that a tensor force acts appreciably in contributing to the binding of ^3H only if the tensor force is longer ranged than the central force. Again turning to meson theory, we find that the tensor force should be longer ranged than the central force in the NN case, but not in the ΛN case. This point is discussed further in reference (7) and (17). Thus, if the predictions of the meson theory are valid, tensor ΛN forces are not important in $^3H_\Lambda$.

Finally we point out that the range assumed for the ΛN potential is actually more appropriate to the NN potential. If a somewhat shorter range, more singular ΛN potential is taken, it should be more effective in binding $^3H_\Lambda$. Thus, the calculated value of s should be reduced. This consideration makes it seem extremely unlikely that the hyperdeuteron can exist, even if $V_{\Lambda s} > V_{\Lambda t}$, provided the form of V_Λ is a central, hard-core potential with a short range tail. However, the fact that equation (13) is on the borderline of satisfying the inequality 12*b*) emphasizes the need for more accurate variational calculations with hard core potentials.

* * *

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RIASSUNTO (*)

Si esegue un calcolo variazionale dell'energia di legame dell'ipertritone allo scopo di determinare l'intensità del potenziale di un nucleone Λ con un nocciolo ripulsivo infinito. Si fa uso di un potenziale di « hard core » anche per descrivere l'interazione protone-neutrone. Si trova un potenziale del nucleone Λ superiore a quello determinato con calcoli precedenti basati sull'ipotesi di potenziali monotonicamente variabili. Nondimeno, il potenziale del nucleone Λ non appare ancora abbastanza intenso per legare l'iperdeutone. Nel calcolo si omettono le forze tensoriali ma si danno argomenti per dimostrare che la loro omissione non influisce sostanzialmente sui risultati.

(*) *Traduzione a cura della Redazione.*

Neutral V-Particle from Copper and Carbon (*).

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G. Petrucci, R. A. Salmeron and A. Zichichi

CERN - Geneva

(ricevuto l'11 Febbraio 1958)

Summary. — The results of a cosmic-ray experiment on the production of neutral V-particles in copper and carbon are given. The conclusions are based on the analysis of 51 V^0-events from copper and 28 from carbon. The ratio of number of Λ^0-particles to number of short-lived θ^0-particles at production is found to be $\sim 1:1$ in copper and $\sim 1:2$ in carbon before allowance is made for the decay modes involving only neutral particles. The momentum spectra found for the V^0-particles agree with earlier results. A possible asymmetry in the decay of Λ^0-particles is discussed and estimates of the mean lifetimes of Λ^0 and θ_1^0-particles are given.

1. – Introduction.

During the past few years it has become clear that cosmic-ray experiments can no longer compete with accelerators in the study of the properties of the known strange particles. The same conclusion does not yet apply to the study of the production processes of the strange particles. Cosmic-ray experiments have the advantage (which is certainly short-lived) that particles are readily available with energies well above the thresholds of all the expected production reactions.

The results described in this paper come from an experiment designed to study the production of strange particles in materials of low and high atomic weight (carbon and copper). The targets were placed successively across the

(*) A part of the contents of this paper was presented at the *International Conference on Mesons and Recently Discovered Particles*, Padua-Venice (September 1957).

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centre of the magnet cloud chamber at the Jungfraujoch [1]. Above the chamber was placed a layer of 30 cm lead. Nuclear interactions in this lead produced a large flux of energetic secondaries which passed through the targets. Seventy-nine neutral V-particles produced in the interactions of these secondaries were observed to decay in the chamber. The analysis of these events is given in Sect. **2** and **3** below.

In making this analysis, an apparent asymmetry in the decay of slow Λ^0-particles was found. This is discussed in Sect. **4**. Finally, the decays were used to make new estimates of the lifetimes of Λ^0- and θ_1^0-particles. These estimates are given in Sect. **5**.

2. – Selection and classification of data.

The experiments using copper and carbon targets were performed consecutively. The copper target was a sheet, 1.2 cm thick, placed centrally across the cloud chamber. The carbon target was made of graphite blocks (density 2.0 g/cm^3) and was 2.5 cm thick. To increase the number of V-events available from carbon, a second target in the form of a wedge (average thickness 5 cm) was placed under the roof of the cloud chamber. For some purposes the V-particles from both carbon targets could be used but for making a comparison between the absolute frequencies of production in copper and carbon only the V-particles coming from interactions in the central plates were considered.

V-events associated with nuclear interactions in the targets were selected. For the neutral V-events the requirements for this association were, in any case, that the interaction should be in the plane of the V-event and, where complete analysis of the V-event was possible, that the calculated line of flight should pass through the interaction within the limits of experimental error. Table I gives a summary of the yield of the experiment.

TABLE I. – *Summary of V-events from Copper and Carbon.*

Target	Number of photographs	Number of V^0-events	Number of $V^{\pm}$-events	Number of pairs of V-events
Copper	35135	51	3	6
Carbon	18877	28	4	2
Total	54012	79	7	8

The table shows the total numbers of V-particles observed to come from interactions in the copper and carbon targets. The copper target was a sheet 1.2 cm thick placed centrally in the cloud chamber. There were two carbon targets, one a central sheet 2.5 cm thick and the second a « wedge » with average thickness 5.0 cm, placed under the roof of the cloud chamber.

[1] J. A. NEWTH: *Suppl. Nuovo Cimento*, **11**, 296 (1954).

The decays were classified as being compatible or incompatible with the process $\Lambda^0 \to p + \pi^- + 37$ MeV. Those events that were not compatible with Λ^0-decay could all be interpreted as decays of the type $\theta_1^0 \to \pi^+ + \pi^- + 214$ MeV. Measurements of momenta, angles and ionization made the classification of most of the events unambiguous but there remained 31 events which could be interpreted either as Λ^0-decays or as θ_1^0-decays. These 31 events were divided statistically using the α-parameter of PODOLANSKI and ARMENTEROS [2] whose magnitude could be determined for every event from angle measurements alone. It was assumed: *a*) that θ_1^0-particles decay isotropically in the centre of mass system with α evenly distributed between the allowed limits, and *b*) that there is symmetry in Λ^0-decays between events where the proton is emitted forwards in the centre of mass system ($\alpha > 0.69$) and events where the proton is emitted backwards ($\alpha < 0.69$) (*).

With these assumptions the division of all the V^0-events into Λ^0- and θ^0-decays can be made; there is, of course, a statistical error in the division based on the distribution of values of α. Table II shows the results of the separation for all the 79 V^0-events that we consider. The division between fast and slow particles at a momentum of 700 MeV/c is made because there is no statistical error in the classification below this momentum, all the decays being unambiguously identified.

TABLE II. – *Analysis of Neutral V-particles.*

Momentum category	Copper		Carbon		Total	
	Λ^0	θ^0	Λ^0	θ^0	Λ^0	θ^0
< 700 MeV/c	13 (32.5)	1 (4.5)	4 (10)	1 (4.5)	17 (42.5)	2 (9)
> 700 MeV/c	13 (19.5)	24 (48)	6 (9)	17 (34)	19 (28.5)	41 (82)
Total	26 (52)	25 (52.5)	10 (19)	18 (38.5)	36 (71)	43 (91)

The table shows how the 79 V^0-events may be subdivided into different categories. For the low-momentum particles ($p < 700$ MeV/c) the separation of Λ^0- and θ^0-decays can be made unambiguously. For the particles with momenta above 700 MeV/c part of the separation has been made directly and part by using a statistical argument.

The figures in parentheses give the number of particles *produced* corresponding to the decays *observed*. In other words, they are obtained by dividing the number of decays observed by the probability of observation; the latter depends on the mean lifetime and momentum of the particles and on the geometry of the cloud chamber. The mean lifetime assumed were $3.0 \cdot 10^{-10}$ s for Λ^0-particles and $1.0 \cdot 10^{-10}$ s for θ^0-particles.

(2) J. PODOLANSKI and R. ARMENTEROS: *Phil. Mag.*, **45**, 13 (1954).

(*) This assumption is discussed in Sect. **4**.

Table II also shows estimates of the number of Λ^0- and θ^0-particles produced in the targets corresponding to the number of decays observed. These estimates are made by weighting the observations in the manner described by GAYTHER and BUTLER (3), using $1/P$ as the weight of each decay event where P is the probability that the decay should have occurred in the gas of the cloud chamber and been observed. The weights depend upon the assumed mean lifetimes of the particles but, in fact, the error due to this uncertainty is only large for the very rare slow θ^0-particles. Implicit in the weighting procedure is the assumption that all the θ^0-particles observed have the same mean lifetime. This point is discussed in Sect. **3** and **5**.

3. – Results and discussion.

3·1. *Momentum spectra at production.* – The most striking feature of the numbers in Table II is the scarcity of θ^0-particles with momenta below 700 MeV/c from both copper and carbon. This is in accord with the results found by JAMES (4) and GAYTHER (5) who studied the production of V^0-particles in lead. There can be no doubt, from all these results, that the differential momentum spectrum of θ^0-particles has a maximum in the neighbourhood of 1 GeV/c and decreases rather rapidly for lower momenta. The integral momentum spectrum of our θ^0-particles follows, above 1 GeV/c, a power law with exponent ~ -1.5.

For the Λ^0-particles, our results are again in agreement with those of JAMES and SALMERON (6) and GAYTHER and BUTLER (3). The differential momentum spectrum of Λ^0-particles rises with decreasing momentum down to momenta of ca. 200 MeV/c (below which value we have no observations). Unfortunately, our data are insufficient to show any difference between the spectra of Λ^0-particles from copper and carbon.

3·2. *The ratio $N(\Lambda^0):N(\theta^0)$.* – Table II shows that the ratio $N(\Lambda^0):N(\theta^0)$ at production is $\sim 1:1.0$ in copper and $\sim 1:2.0$ in carbon. This difference is significant statistically though, of course, the true difference could be less than what is observed. The ratios apply to V^0-particles with momenta between 200 MeV/c and about 3 GeV/c that are emitted near to the vertical. Outside these limits the efficiency of detection in our apparatus is small.

If we assume that the production of Λ^0-particles per atom of different target materials follows a law of the form $N(\Lambda^0) \propto A^x$ and that for θ^0-part-

(3) D. B. GAYTHER and C. C. BUTLER: *Phil. Mag.*, **46**, 467 (1955).
(4) G. D. JAMES: *Suppl. Nuovo Cimento*, **4**, 325 (1956).
(5) D. B. GAYTHER: *Phil. Mag.*, **45**, 570 (1954); **46**, 1362 (1955).
(6) G. D. JAMES and R. A. SALMERON: *Phil. Mag.*, **46**, 571 (1955).

icles a similar law $N(\theta^0) \propto A^y$, the ratio $N(\Lambda^0)/N(\theta^0)$ will vary as A^{x-y}. From our observations the difference $x-y$ is equal to 0.42 with an asymmetric statistical error which can best be represented by saying that there is 95% probability that the true value is between 0.20 and 0.65.

3·3. *Probabilities of production in carbon and copper.* – In the above argument there is no need to have an exact knowledge of the target thicknesses since only ratios are compared for the two target materials. However, if we wish to determine the separate values of x and y, rather than $x-y$, the target thicknesses are of primary importance and the only V^0-particles from carbon that we may include are those produced in the central plate across the cloud chamber. Since these form only half of the total numbers listed in Table II, the statistical accuracy is poor. However, the calculated values (numbers per atom) are the following:

$$\text{if} \quad N(\Lambda^0) \propto A^x \quad \text{then} \quad 1.0 < x < 1.5$$

and

$$\text{if} \quad N(\theta^0) \propto A^y \quad \text{then} \quad 0.6 < y < 1.0\,.$$

These results cannot be directly translated into cross-sections since the flux of primary particles producing the V^0-particles is not homogeneous. It is reasonable to assume that the majority are π-mesons with momenta of several GeV/c, but there are certainly some nucleons and K-mesons in addition.

3·4. *The effect of unobservable decays.* – All the above results apply the to production of short-lived particles whose decays can be recognized. If there are, in addition, Λ^0- and θ^0-particles that are very long-lived or which decay into neutral secondaries, the figures for production given in Table II must be correspondingly increased.

On this subject, the most valuable work available is that of EISLER *et al.* (7) and the observations reported by CHINOSKY *et al.* (3). EISLER *et al.* find that the decay $\Lambda^0 \to n + \pi^0$ takes place for about one-third of all Λ^0-particles. The figures in Table II referring to the production of Λ^0-particles must be multiplied by 1.5 to take account of these neutral decays.

For the θ^0-particles there is now considerable experimental evidence that confirms the theoretical prediction that half the particles should decay with

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(7) F. EISLER, R. PLANO, N. SAMIOS, M. SCHWARTZ and J. STEINBERGER: *Nuovo Cimento*, **5**, 1700 (1957).

(8) W. CHINOSKY, K. LANDÉ and L. M. LEDERMANN: *International Conference on Mesons and Recently Discovered Particles*, Padua-Venice (September 1957).

a long mean lifetime (θ_2^0-decays). Moreover, some of the short-lived θ_1^0-particles should decay into two π^0-mesons (about 12% according to EISLER *et al.*). For these reasons, the numbers of θ^0-mesons given in Table II have to be multiplied by a factor of about 2.3.

The overall result of these changes is to alter the ratios of $N(\Lambda^0):N(\theta^0)$ at production so that they become $\sim 1:1.5$ in copper and $\sim 1:3.0$ in carbon.

3·5. *Discussion.* – In our present experiment we find a significant difference between the values of the ratio $N(\Lambda^0):N(\theta^0)$ for production in carbon and copper. The values of the ratio are 1 : 2 for carbon and 1 : 1 for copper before allowance is made for unobservable decays. GAYTHER [5] in a comparable cosmic-ray experiment found a ratio of about 1 : 0.5 for production in a lead target.

Using π -mesons with an energy of 1.9 GeV from the Cosmotron, BLUMENFELD *et al.* [9,10] studied the production of V^0-particles in carbon and lead. The ratio $N(\Lambda^0):N(\theta^0)$ at production was found to be 1 : 0.8 in carbon and 1 : 0.26 in lead.

From these different results two qualitative conclusions can be drawn. First, that in both cosmic-ray and machine experiments the ratio $N(\Lambda^0):N(\theta^0)$ increases with increasing size of the target nucleus. Second, for a given target material, the ratio $N(\Lambda^0):N(\theta^0)$ is higher in the machine experiment than in the cosmic ray experiment. The data are not adequate for these two conclusions to be expressed in a valuable quantitative way.

The increase in the ratio $N(\Lambda^0):N(\theta^0)$ in heavy nuclei can be explained by more Λ^0-particles than θ^0-particles being produced in secondary reactions in the nucleus or by more θ^0-particles than Λ^0-particles being absorbed or by some combination of the two effects. However, $N(\theta^0)$ alone varies approximately as $A^{\frac{2}{3}}$ and this is what would be expected if the production of θ^0-particles occurred mainly in « primary » collisions of particles incident on the target and if there were little subsequent absorption of the θ^0-particles in the parent nucleus. With this over-simplified picture it is possible to explain the values of $N(\Lambda^0):N(\theta^0)$ by considering only those secondary reactions that lead to the production of additional Λ^0-particles.

BLUMENFELD *et al.* proposed this type of explanation of their results. In addition to the direct production of Λ^0-particles in π^--proton collisions they considered that the reactions

$$\pi^- + \mathrm{N} \rightarrow \Sigma^- + \mathrm{K} \tag{1}$$

([9]) H. BLUMENFELD, E. T. BOOTH, L. M. LEDERMANN and W. CHINOWSKY: *Bull. Amer. Phys. Soc.*, Ser. II, 1, 63 (1956).

([10]) H. BLUMENFELD: *Ph. D. Thesis* (Columbia University, 1957).

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could be followed by

$$\Sigma^- + p \to \Lambda^0 + n \tag{2}$$

occurring in the same nucleus.

At cosmic-ray energies the allowed production processes of strange particles are more numerous and, in particular, there is good reason to believe that the production of pairs of K-mesons is an important process ([11,12]). The sequence of reactions leading to the formation of Λ^0-particles in heavy nuclei can then be

$$\pi + N \to N + K + \overline{K} \tag{3}$$

followed by

$$\overline{K} + N \to \Lambda^0 + \pi \,. \tag{4}$$

If reaction (3) leads to a $\overline{K}$-meson of sufficiently high energy, a further interesting possibility arises. This is that the K-meson interacts with a nucleon to form a Ξ-particle which subsequently decays into a Λ^0-particle. The processes are

$$\overline{K} + N \to \Xi + K \tag{5}$$

and

$$\Xi \to \Lambda^0 + \pi \,. \tag{6}$$

The interest of this process lies in the fact that the Λ^0-particle is produced in a « slow » decay where « strangeness » is not conserved. This is discussed further in Sect. **4**.

The difference between the values for the ratio $N(\Lambda^0):N(\theta^0)$ in carbon found by us and by BLUMENFELD *et al.* can readily be explained. The π^--mesons used by BLUMENFELD *et al.* had an energy only slightly above the threshold for reaction (3). If the production of K-meson pairs is a process that occurs frequently at cosmic-ray energies, it provides a mechanism by which θ^0-particles may be produced unaccompanied by hyperons. Our result may therefore be taken as further evidence for the great importance of the pair production of K-mesons in cosmic ray experiments.

Unfortunately, the interpretation is complicated by the presence of charged K-mesons among the primary particles that produce nuclear interactions in

(11) W. A. COOPER, H. FILTHUTH, J. A. NEWTH and R. A. SALMERON: *Nuovo Cimento*, **4**, 390 (1956).

(12) W. A. COOPER, H. FILTHUTH, J. A. NEWTH, G. PETRUCCI, R. A. SALMERON and A. ZICHICHI: *Nuovo Cimento*, **5**, 1388 (1957).

our targets. The positive excess among cosmic-ray K-mesons may result in the production of more θ^0- than Λ^0-particles since only negative K-mesons are expected to have a high cross-section for producing Λ^0-particles while both positive and negative K-mesons should produce θ^0-particles. Not enough is known about the interactions of energetic K-mesons for the exact importance of this effect to be calculated but it could not, alone, explain the difference between our result and that of BLUMENFELD *et al.*

4. – A possible asymmetry in the decay of Λ^0-particles.

In Sect. 2, the distribution of the unidentified V^0-events between Λ^0-decays and θ^0-decays was described. It depends, essentially on the assumption that the proton in Λ^0-decay is not emitted preferentially forwards or backwards in the centre of mass system. Expressed differently, if θ^* is the angle in the centre of mass system that the proton trajectory makes with the direction of motion of the Λ^0-particle in the laboratory system, it is assumed that $\cos\theta^*$ is equally likely to be positive or negative.

The only theoretical argument in favour of this symmetry applies to Λ^0-particles produced in parity-conserving reactions ([13]) but it is not known how many cosmic-ray Λ^0-particles are produced in such reactions. For example, Λ^0-particles can be produced in the decay of Ξ-particles where parity may not be conserved.

In our experiment we found 17 identified, slow Λ^0-particles and among the decays of these particles there are 3 with positive values of $\cos\theta^*$, 11 with negative values and 3 where the value of θ^* is near 90° and $\cos\theta^*$ is of doubtful sign. There is no bias involved in selecting these 17 events since not one of our unidentified V^0-decays can be interpreted as a Λ^0-decay with momentum less than 700 MeV/c.

This indication of a lack of symmetry led us to investigate the existing experimental evidence more carefully. There are two earlier cosmic-ray experiments where this symmetry has been specifically studied. ARMENTEROS ([14]) first called attention to the problem. He reported that among 18 events, 6 gave positive values of $\cos\theta^*$ and 12 gave negative values. GAYTHER ([5]) found a symmetrical distribution of $\cos\theta^*$ (11 positive and 10 negative values). Some further information can be found in tables of measurements published in connection with two other experiments ([15,16]) which both show an excess

([13]) T. D. LEE and C. N. YANG: *Phys. Rev.*, **104**, 254 (1956).

([14]) R. ARMENTEROS: *Report of the Bagnères Congress on Cosmic Radiation* (1953).

([15]) H. S. BRIDGE, C. PEYROU, B. ROSSI and R. SAFFORD: *Phys. Rev.*, **91**, 362 (1953).

([16]) W. B. FRETTER, M. M. MAY and M. P. NAKADA: *Phys. Rev.*, **89**, 168 (1953).

of negative values of $\cos\theta^*$. However, there is certainly an observational bias that favours the recognition of Λ^0-decays with negative values of $\cos\theta^*$ and this bias can only be avoided by considering the very slowest Λ^0-decays (with momenta less than 600 MeV/c). In such decays the proton, even when emitted forwards, has an ionization of at least 2.5 times the minimum value. In the two experiments considered there are 28 decays of Λ^0-particles with momenta below 600 MeV/c. These give 8 positive and 18 negative values of $\cos\theta^*$; 2 are of doubtful sign.

In addition to the above cosmic-ray experiments there is a machine experiment of BUDDE *et al.* ([17]) in which the decays of Λ^0-particles produced in the collisions of 1.3 GeV π^--mesons with protons were studied. In this experiment no asymmetry of the type we discuss was found (11 positive values of $\cos\theta^*$ and 10 negative values). This result is not strictly comparable with the cosmic-ray data concerning Λ^0-particles produced either directly or indirectly in high-energy interactions in complex nuclei.

To obtain more information we decided to make a further independent determination of the asymmetry using some photographs, not previously analyzed, showing the decays of slow Λ^0-particles produced in lead. 30 Λ^0-decays were found by selecting V^0-events where the positive secondary particle could be identified as a proton from its momentum and ionization. To reduce the observational bias already mentioned we considered only 21 Λ^0-particles with momenta below 600 MeV/c. In the decays of these particles there were 8 positive values of $\cos\theta^*$, 10 negative and 3 of doubtful sign.

If we combine together the results from the cosmic-ray experiments where the symmetry of Λ^0-decay has been specifically considered (and where, presumably, the residual bias is small) we find that the values of $\cos\theta^*$ are distributed as 28 positive, 43 negative and 6 of doubtful sign. If we add the results of the two further experiments ([15,16]), the distribution becomes 36 positive, 61 negative and 8 of doubtful sign. The probabilities for such asymmetric distributions to arise by statistical fluctuation from a symmetrical population are 7.5% and 1.1% respectively.

To sum up, there is some evidence for the existence of a forward-backward asymmetry in the decay of slow cosmic-ray Λ^0-particles. The evidence is not conclusive statistically and there is insufficient knowledge of the effects of bias in scanning and recognition of the events in some of the experiments for us to be able to accept the data at their face value. The evidence for an asymmetry—and the importance of the effect—are certainly great enough to justify further experiment.

([17]) R. BUDDE, M. CHRÉTIEN, J. LEITNER, N. P. SAMIOS, M. SCHWARTZ and J. STEINBERGER: *Phys. Rev.*, **103**, 1827 (1956).

5. – Mean lifetimes of Λ^0- and θ_1^0-particles.

Having available a fairly large number of Λ^0- and θ_1^0-decays it was a relatively simple matter to use these events to determine the mean lifetimes of the two types of particle. The techniques used for this determination are well known [18,19].

5·1. *Λ^0-particles.* – Using 21 Λ^0-particles produced in the copper and carbon targets placed inside the cloud chamber, the calculation gave the following mean lifetime and statistical error:

$$\tau_{\Lambda^0} = (2.86^{+1.14}_{-0.59})\cdot 10^{-10}\ \text{s}\ . \ (*)$$

Including in the calculation also the 19 Λ^0-particles referred to in Sect. **4**, we find that

$$\tau_{\Lambda^0} = (3.04^{+0.78}_{-0.51})\cdot 10^{-10}\ \text{s}\ .$$

5·2. *θ_1^0-particles.* – Using 29 θ_1^0-decays we find for the mean lifetime of these particles:

$$\tau_{\theta_1^0} = (0.84^{+0.35}_{-0.19})\cdot 10^{-10}\ \text{s}\ .$$

In connection with the mean lifetime of the θ_1^0-mesons there are two comments that should be made. First, one requirement of the selected decays was that the plane of the V-event should contain, within the experimental error, the nuclear interaction from which the V-particle was presumed to come. This requirement reduces the chance that a θ^0-meson, decaying into three secondaries, should be included in the analysis. In other words, the mean lifetime that has been estimated should be based almost entirely on decays of the type $\theta_1^0 \to \pi^+ + \pi^-$. To calculate the momenta of the θ^0-particles, where these could not be directly measured, the θ_1^0-decay scheme was assumed.

The second comment is that one single event among the 29 has a very great influence on the result. This θ^0-particle has a lifetime before decay of $3.3\cdot 10^{-10}$ s and its inclusion raises the estimate of the mean lifetime from $0.63\cdot 10^{-10}$ s to the value quoted above. The decay is well measured and ap-

(18) M. S. Bartlett: *Phil. Mag.*, **44**, 249 (1953).

(19) J. A. Newth: *Proc. Roy. Soc.*, A **221**, 406 (1954).

(*) The Λ^0-lifetime found from the combined results of three machine experiments is: (7)(10)(20)$\tau_{\Lambda^0} = (2.75 \pm 0.17)\cdot 10^{-10}$ s.

(20) J. L. Brown, D. A. Glaser and M. L. Perl: preprint.

pears to be of the type $\theta_1^0 \to \pi^+ + \pi^-$ (*). Whether it is a θ_1^0-decay or a θ_2^0-decay that is dynamically similar, we cannot say but, in any case, the event serves to show that the uncertainty in our knowledge of the mean lifetime of the θ_1^0-particles is greater than is indicated by the statistical error alone.

* * *

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Dr. C. VERKERK assisted with the determinations of mean lifetimes reported in Sect. **5**. For this, we are extremely grateful.

We have been fortunate in having opportunities to discuss several parts of our work with Prof. H. S. BRIDGE, Dr. B. D'ESPAGNAT and Dr. J. PRENTKI.

(*) The measured quantities were the momentum of the positive track and the angles of the two secondaries with the line of flight of the θ^0-particle.

RIASSUNTO

È stata studiata la produzione di particelle V neutre in interazioni nucleari di raggi cosmici con rame e carbonio. Le conclusioni sono basate sull'analisi di 51 V^0 da rame e 28 da carbonio. Il rapporto, non corretto per decadimenti neutri, tra il numero di particelle Λ^0 e quello delle particelle θ^0, a breve vita media, risulta essere alla produzione: 1/1 per gli eventi da rame e 1/2 per quelli da carbonio. Gli spettri dei momenti trovati per le V^0 sono in accordo con precedenti determinazioni. Si discute la possibilità dell'esistenza di una asimmetria nel decadimento delle particelle Λ^0 e vengono dati i valori delle vite medie per le Λ^0 e θ_1^0.

Inner Bremsstrahlung in Mu-Meson Decay.

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(ricevuto il 27 Febbraio 1958)

Summary. — The transition probability for decay of the μ-meson with the emission of an additional photon is computed on the basis of the most general two-component theory of the neutrino with lepton conservation. The estimated ratio of one radiative event per 10^5 decays suggests that the experiment may be feasible in the near future.

In the older literature on the weak decays, considerable interest was manifested in the study of extra photon production accompanying charged particle transformation as a possible means of obtaining information about the coupling constants which determine the non-radiative processes. In particular, the emission of a photon in the decay of the μ-meson was discussed exhaustively by Lenard (1). The recent convergence in our knowledge of the weak interactions, in particular the rather dramatic reemergence of a universal coupling in a hitherto unexpected form (2) has reduced the intrinsic interest in the study of the electromagnetic processes. On the other hand, the development of high flux accelerators together with the promised development of the counting techniques with essentially complete solid angle acceptance will shortly make such experiments feasible for the first time.

We initially consider the most general form of the two component neutrino theory with lepton conservation specializing below to the universal form of the interaction. The couplings involved are defined by the following matrix

(1) A. Lenard: *Phys. Rev.*, **90**, 968 (1953).

(2) R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958); E. C. G. Sudarshan and R. E. Marshak: *Bull. Amer. Phys. Soc.*, **3**, no. 1, 20 (1958).

element for the non-radiative decay of the μ^-:

$$(1)\qquad M = \bar{u}(p)\left\{g_1\gamma_\mu \tfrac{1}{2}(1 - i\gamma_5) + g_2\gamma_\mu \tfrac{1}{2}(1 + i\gamma_5)\right\} u(P)\, \bar{u}(q)\tfrac{1}{2}\gamma_\mu(1 + i\gamma_5)v(q)\,,$$

where p, P, q, $\bar{q}$ are four-momenta of electron, muon, neutrino, and antineutrino respectively. For the process of inner bremsstrahlung, we then obtain the following transition probability for a μ-meson at rest and polarized in the z direction:

$$(2)\qquad W = \frac{\alpha\pi}{6k}\frac{1}{(2\pi)^{10}}\, d^3p\, d^3k\, [|g_1|^2 + |g_2|^2] f_0(p, k, \boldsymbol{\sigma}_\mu, \boldsymbol{e}) + \\ + [|g_1|^2 - |g_2|^2] f_3(p, k, \boldsymbol{\sigma}_\mu, \boldsymbol{e})\,,$$

where α is the fine structure constant, k the photon momentum, $\boldsymbol{e}$ its polarization vector, $\boldsymbol{\sigma}_\mu$ a unit vector in the direction of the μ-meson spin, and where the functions $f_{0,3}$ are given by the expressions

$$(3)\qquad f_i = \frac{(ep)^2}{(kp)^2}\left\{2(pG)G_i + (GG)p_i + \frac{(pk)}{M}[2(Gp) + 2(Gk) + 2MG_i - (GG)] + \right. \\ \left. + 2(Gk)G_i + (GG)k_i + \frac{2}{M}(pk)^2\right\} + \left\{2(GG) - \frac{2(pG)(kG)}{Mk}\right\}\frac{(ep)}{(kp)}e_i + \\ + \frac{(GG)}{M}4\frac{k_3}{k} + \frac{(kG)^2}{M^2k(pk)}[2Mp_i + 2MG_i - 2(pG)] + \\ + \frac{(GG)(kG)}{M^2k(pk)}[Mk_i - (pk)] + [(Gk) - 2(pk)]\frac{(ep)}{Mk(pk)}e_i\,.$$

Here i takes on the values, $i = 0.\,3$; (AB) means $A_0B_0 - \boldsymbol{A}\cdot\boldsymbol{B}$, and

$$(4)\qquad G_\mu = [P - p - k]_\mu\,,$$

that is,

$$(5)\qquad \boldsymbol{G} = -\boldsymbol{p} - \boldsymbol{k}\,, \qquad G_0 = M - p - k\,.$$

Since one will undoubtedly detect the photons by means of their subsequent pair production, there exists some possibility of obtaining information about photon polarization as well as of studying the correlation between the directions of the electron and the photon.

If one is not interested in the photon polarization, then the following replacements are in order:

$$(6)\qquad (ep)^2/(kp)^2 \to [p^2 - (p\cdot k)^2/k^2]/(kp)^2\,, \qquad (ep)e_3/(kp) \to [p_3 - (p\cdot k)k_3/k^2]/(kp)\,.$$

It is perhaps worth noting a simple argument for the occurrence in W of the combinations $|g_1|^2+|g_2|^2$ and $|g_1|^2-|g_2|^2$ alone. The Hamiltonian corresponding to Eq. (1), for example, is invariant under the transformation of electron variables

$$\psi \to i\gamma_5\psi\,, \qquad \overline{\psi} \to \overline{\psi}(-i\gamma_5)\,, \qquad m \to -m\,, \tag{7}$$

with the simultaneous replacements

$$g_1 \to -g_1\,, \qquad g_2 \to g_2\,. \tag{8}$$

Thus the other possible independent combinations $\mathrm{Re}\, g_1 g_2^*$ and $\mathrm{Im}\, g_1 g_2^*$ of g_1 and g_2 can occur only in terms proportional to m, which we neglect.

The universal form of coupling is obtained by setting $g_2=0$, $g_1=(8)^{\frac{1}{2}}G$ in the notation of FEYNMAN and GELL-MANN (2).

From Eqs. (2) and (3) in conjunction with the corresponding equation for non-radiative decay (3), we estimate the relative probability for inner bremsstrahlung to be 10^{-5} per μ-decay. Essentially the same result follows from simple phase space considerations.

* * *

We wish to thank Dr. W. SELOVE for an interesting discussion of the experimental possibilities.

(3) T. D. LEE and C. N. YANG: *Phys. Rev.*, **108**, 1611 (1957).

RIASSUNTO (*)

Si calcola la probabilità di transizione per decadimento del mesone μ con l'emissione di un fotone addizionale sulla base della più generale teoria del neutrino a due componenti con conservazione dei leptoni. La proporzione prevista di un evento radiativo per 10^5 decadimenti suggerisce che in un prossimo futuro l'esperienza sarà eseguibile.

(*) *Traduzione a cura della Redazione.*

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On the Nucleon-Antinucleon Interactions (*).

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Résumé. — Les sections efficaces nucléon-antinucléon ont été calculées à 167 MeV avec deux modèles à puits de potentiel complexe. On a d'abord étudié des puits complexes de Yukawa, dans l'approximation de Born. Un potentiel semi-théorique a ensuite été analysé sur l'Ordinateur électronique IBM 704: la partie réelle est le potentiel de Signell et Marshak, adapté au problème nucléon-antinucléon, et la partie imaginaire, un puits phénoménologique de Yukawa. Pour rendre compte des résultats expérimentaux (grande section efficace totale et petite section efficace de diffusion), il apparaît nécessaire que le potentiel réel soit faible et que le potentiel imaginaire ait une portée au moins égale à la portée usuelle des forces nucléaires. La théorie mésique de la source fixe d'une part conduit à un potentiel réel trop fort, et d'autre part ne prédit pas de potentiel imaginaire; cette théorie ne peut pas rendre compte des résultats expérimentaux.

1. – Introduction.

The experiments (1) on the nucleon-antinucleon cross-sections, although they are still in a preliminary stage, exhibit two striking features which deserve theoretical understanding:

1) For a given energy, the total cross-sections are much larger than in the nucleon-nucleon case: at 190 MeV in the laboratory system, the anti-

(*) Supported in part by the United States Air Force through the European Office Air Research and Development Command.

(1) O. Chamberlain, E. Segrè, C. Wiegand and T. Ypsilantis: *Phys. Rev.*, **100**, 947 (1955); B. Cork, G. R. Lambertson, O. Piccioni and W. A. Wentzel: *Phys. Rev.*, **104**, 1193 (1956); **107**, 248 (1957); O. Chamberlain, D. V. Keller, R. Mermod, E. Segrè, H. H. Steiner and T. Ypsilantis: *Phys. Rev.*, **108**, 1553 (1957).

proton-proton cross-section is of the order of 135 millibarns, in contrast with the 22 mb obtained with protons of the same energy.

2) The total cross-section seems to be essentially due to annihilation. If we choose to believe the experiments in their present state, there seems to be little room for elastic and exchange scattering, which contribute probably less than 20% of the total cross-section [2].

The large total cross-section has been explained by BALL and CHEW [3] using a model inspired by the success of the fixed source meson theory. This theory leads, in the nucleon-nucleon case, to the Gartenhaus potential [4]; when a phenomenological spin-orbit term is added [5], it gives a good fit to the scattering and polarization data up to 150 MeV. A nucleon-antinucleon interaction can be obtained by reversing the sign of the second order term of the Gartenhaus potential, to take into account the charge conjugation between nucleon and antinucleon. BALL and CHEW make the two further assumptions:

1) The Signell-Marshak spin-orbit term remains unchanged when one goes to the nucleon-antinucleon case.

2) The inner region of the interaction ($r < 0.4(\hbar/\mu c)$, where μ is the π-meson mass), of which little can be said from meson theory, is replaced by an absorbing core in which the annihilation takes place. The total cross-section at 140 MeV, computed in W.K.B. approximation with such an interaction is in good agreement with experiment. The reason why the total cross-section is larger for antinucleons than for nucleons seems to be that, in the latter case, there is a cancellation between 2nd and 4th order terms. This cancellation disappears in the antinucleon problem where the sign of the 2nd order term has been changed.

The small elastic and exchange cross-sections are much more difficult to explain. The absorption predicted by CHEW and BALL is quite small (69 mb). As we shall see later, an exact calculation with the same model leads to even smaller results. This is in definite contradiction with the experiments if we choose to believe them in their present state.

Similar results were obtained by KOBA and TAKEDA [6], who also considered a small black sphere, surrounded by a real square well.

[2] O. CHAMBERLAIN: private communication.
[3] J. S. BALL and G. F. CHEW: UCRL-3922 (1957); *Phys. Rev.*, **109**, 1385 (1958).
[4] S. GARTENHAUS: *Phys. Rev.*, **100**, 900 (1955).
[5] P. S. SIGNELL and R. E. MARSHAK: *Phys. Rev.*, **106**, 832 (1957).
[6] Z. KOBA and G. TAKEDA: to be published.

Actually, the failure of the above models to account for a small scattering cross-section is not surprising, and should be related to the choice of ranges which are too small, especially as far as the absorbing region is concerned. It has been shown, in fact, by very general arguments [7] that, in order to obtain a small scattering cross-section together with a large total cross-section, it is necessary to have many partial waves, and thus a large extension of the absorbing potential.

We shall therefore introduce, in a phenomenological way, a long range imaginary potential. Some theoretical argument has been put forward by M. Lévy [8] in favour of such a potential. In a preliminary exploration (Sect. 2), we investigate the scattering and annihilation cross-sections obtained with simple real and imaginary Yukawa potentials, having independent depths and ranges. Agreement with experiment requires that the imaginary potential has a range at least of the order of the meson Compton wave length, and that the real potential has a relatively small depth.

In a following Section (3), we try to improve Ball and Chew's model by adding to their real potential a phenomenological Yukawa shaped imaginary potential. An exact phase-shift analysis leads us to the conclusion that the fixed source meson theory in its present state provides a real potential which is much too strong to account for the small scattering cross-sections.

2. – Complex potential with Yukawa wells.

In this Section, we intend to calculate in Born approximation, the scattering and absorption cross-sections with the complex potential:

$$V(r) = -V_R \frac{\exp[-\alpha r]}{\alpha r} - iV_I \frac{\exp[-\beta r]}{\beta r}.$$

The choice of a Yukawa shape, at least for the imaginary part, is related to the physical assumption that the absorption should be strong at short distances. The Yukawa shape has also the mathematical advantage that the Born approximation can be calculated analytically up to the second order [9]; the necessary formulae are given in Appendix I. It can be seen that, with our choice of parameters, the convergence of the Born series is good. For a fixed total cross-section, the elastic scattering increases when the ranges α^{-1} and

(7) M. Lévy: *Padua-Venice Conference* (1957); W. Rarita: *Bull. Amer. Phys. Soc.*, **2**, 354 (1957).

(8) M. Lévy: to be published in *Nuovo Cimento*.

(9) P. Morse and H. Feshbach: *Methods of Theoretical Physics* (New York, 1953).

β^{-1} decrease. This result confirms the need for long range potentials in order to fit the small elastic to total cross-section ratio.

We proceed to a more quantitative study for the incident wave number $k = 2\mu$. This wave number corresponds to an incident antiproton energy of 167 MeV in the laboratory system. We consider several sets of values for the ranges α^{-1} and β^{-1}, and, of each of them, first choose $V_R = 0$. V_I is adjusted so as to yield 120 mb for the corresponding total cross-section. We then vary V_R and look for the maximum value of V_R which is allowed if there is to be no more than 30 mb for the elastic cross-section. The results are in Table I.

TABLE I. – *The elastic and total cross-sections as computed with complex Yukawa wells.*

$\beta^{-1}\mu$	$\alpha^{-1}\mu$	V_I (MeV)	V_R (MeV)	σ_{tot} (mb)	σ_{el} (mb)
0.83	—	30.2	0	120	30
1	—	23.0	0	120	17.9
1	1	23.0	18.9	139.5	30
1	2	23.0	4.83	137.2	30
2	—	2.58	0	120	3.8
2	1	2.58	27.9	149.5	30
2	2	2.58	6.82	156.1	30

It is seen, even without any real potential, that the imaginary part must have at least a range of $0.83\,\mu^{-1}$. The presence of a real potential still increases the scattering. The maximum depth for a real potential of normal range ($\alpha^{-1}\mu = 1$) is rather small (*). Furthermore, if an attractive real potential is able to increase somewhat the annihilation (total minus elastic) cross-sections, at the same time, the elastic cross-section increases by a comparable amount, and the situation is not improved.

We thus have a strong indication that, in order to fit the experimental results, we need a long range imaginary potential (to provide the large total cross-section) and a small real potential (to obtain a small scattering cross-section).

3. – Semi-theorical complex potential.

The phenomenological real potentials established in the preceding section are much weaker than the Ball and Chew real potential. Thus, it can be anticipated that the latter will give too much scattering. The results with this interaction, however, cannot safely be guessed from Born approximation.

(*) For the sake of comparison, we recall that the depth of the Yukawa well of the same range which fits the deuteron is 47 MeV.

Furthermore, since the Signell and Marshak potential worked so nicely in the proton-proton case, it is probably worthwhile to attempt to consider it in the present antinucleon problem.

Consequently, we first investigate the following model: the real part of the potential, at least for distances larger than $r_c = 0.38\mu^{-1}$, will be the Signell and Marshak potential, with the sign of the 2nd order term changed, as in the model of Chew and Ball. To this real potential, we add the same phenomenological imaginary Yukawa shaped potential as in Sect. 2. We try to adjust the range β^{-1} and the depth V_I so as to obtain a total cross-section of about 150 mb, and an annihilation cross-section not much smaller, for an incident energy of 167 MeV.

Various models for the core region have been used. Either we describe it by a complex well, the depths of which have been extensively varied in signs and magnitudes, or we simulate the core by absorption boundary conditions at r_c.

In contrast with the nucleon-nucleon system, in the nucleon-antinucleon case we must consider, for given L and S, both isotopic spins 0 and 1. From the corresponding scattering amplitudes M_0 and M_1 one obtains the differential cross-sections for the various processes ([10]):

$$\frac{d\sigma_{p+\bar{p}\to p+\bar{p}}}{d\Omega} = \left|\frac{M_0 + M_1}{2}\right|^2,$$

$$\frac{d\sigma_{p+\bar{p}\to n+\bar{n}}}{d\Omega} = \left|\frac{M_0 - M_1}{2}\right|^2,$$

$$\frac{d\sigma_{n+\bar{p}\to n+\bar{p}}}{d\Omega} = |M_1|^2 .$$

For each partial wave, the computation of M_0 and M_1 is performed with the complex phase-shift method ([11]) for tensor forces, extended to the situation where there is also a complex potential. In the cases of a singlet state and of a triplet $L = J$ state, the real and the imaginary parts of the radial wave functions satisfy two differential equations coupled through the imaginary part of the potential. For the other triplet states, the wave functions are also coupled by the tensor force, and we have a system of four coupled real differential equations. The principal formulae are given in Appendix II.

The calculation of explicit solutions of these differential equations, the determination of proper combinations obeying the various boundary conditions at the core radius, the calculation of the corresponding phase-shifts,

([10]) H. A. Bethe and J. Hamilton: *Nuovo Cimento*, **4**, 1 (1956).
([11]) W. Rarita and J. Schwinger: *Phys. Rev.*, **59**, 436, 556 (1941).

and finally the computation of the cross-sections have been carried out on the IBM 704 digital computer. For $L > 3$, the contributions of the real potential are negligible, and we used the second order Born approximation for the imaginary potential.

The total cross-section for the $p+\bar{p}$ reactions is the sum of three terms:

a) elastic cross-section σ_{el} (final state $p+\bar{p}$);

b) exchange cross-section σ_{ex} (final state $n+\bar{n}$);

c) annihilation cross-section σ_{an}.

The results for various values of the depth and of the range of the imaginary Yukawa potential are tabulated in Table II. The ratio $\varrho = \sigma_{an}/\sigma_{total}$ is also presented. The calculations were performed with 15 different boundary conditions at the core, in order to test the sensitivity of the results to the structure of the core. We only report three sets of results corresponding respectively to the black sphere, to the most favourable case, and to the least favourable one.

The condition I, which is of the same kind as in BALL and CHEW, is such that there are only ingoing waves at the surface of the core, where the logarithmic derivatives of the reduced wave function must be $-ik$ (k is chosen here as the wave number at infinity). For condition II, the core is a complex square well with the parameters $V_R = 650$ MeV, $V_I = 1000$ MeV. Condition III corresponds to a real infinite repulsive core.

The different cross-sections should in principle depend on the initial conditions. But, in the presence of an imaginary potential of sufficient importance, the wave function is damped before it reaches the core. Since the phenomena in this region are little known, it is most satisfactory to obtain results which are fairly independent of the boundary conditions.

The elastic cross-section is practically independent of the presence of an imaginary potential. It essentially comes from the real part of the potential and thus it appears that it is impossible to obtain a small scattering cross-section.

We cannot give serious consideration to the results concerning the exchange cross-section because the latter proceeds from the difference between the amplitudes M_0 and M_1 whereas, in our phenomenological model, the imaginary potential does not depend on the isotopic spin. Nevertheless, the exchange cross-section also seems to be too large, since experimentally it is only of the order of a few millibarns.

The annihilation cross-section is extremely sensitive to the imaginary potential. If we want to obtain a large ratio of annihilation to total cross-section, we must use a strong imaginary potential. As a consequence, the total cross-section becomes much too large. On the other hand, if we adjust the depth of the imaginary potential in order to obtain a reasonable total cross-section,

TABLE II. – *Different cross-sections as computed with the semi-theoretical complex potential.*

Range β^{-1} (μm)	Imaginary depth V_I (MeV)	Initial con-ditions	$\bar{p}$-p cross-sections (mb)					$\bar{p}$-n cross-sections (mb)			
			elastic	ex-change	annihil-ation	total	ϱ = annihil-ation/total	elastic	annihil-ation	total	ϱ = annihil-ation/total
2	10	I	80	14	391	485	.81	64	396	460	.86
		II	75	13	394	482	.82	64	397	461	.86
		III	74	17	386	477	.81	63	397	460	.86
	3	I	83	33	150	266	.56	112	160	272	.59
		II	68	23	160	251	.64	79	171	250	.68
		III	50	34	128	212	.60	49	132	181	.73
	1	I	83	38	70	191	.37	118	80	198	.40
		II	66	27	83	176	.47	80	94	174	.54
		III	48	40	44	132	.33	51	46	97	.47
1	7	I	82	31	73	186	.39	108	83	191	.43
		II	68	22	83	173	.48	77	94	171	.55
		III	50	33	52	135	.39	47	57	104	.55
	0	I	83	42	29	154	.19	123	39	162	.24
		II	66	29	42	137	.31	81	54	135	.40
(*a*) 2	3	II	80	25	158	263	.60	109	171	280	.61
(*b*) 2	1.5	II	50	17	102	169	.60	55	114	169	.67

a) The sign of the spin orbit term is reversed. *b*) The depth of the real potential is multiplied through a factor 0.7.

the maximum value of the ratio $\varrho = \sigma_{an}/\sigma_{tot}$ is 0.5. The experimental ratio is at present much larger.

In addition, we have performed the calculations without any imaginary potential. With condition I, we then have the Ball and Chew model. The exact calculation then gives somewhat less favourable results than those obtained with the W.K.B. approximation. The ratio σ_{an}/σ_{tot} is .19 for $\bar{p}$-p, in strong disagreement with the experimental results. Even with condition II, we cannot obtain a better ratio than .31. The results for the $\bar{p}$-n case are quite similar.

As the spin-orbit term of the potential is a phenomenological one, a run was made with its sign changed. The results are not significantly altered.

Since the real potential seems too strong, a run was made in which it is damped by a factor 0.7. It appears that this reduction is not sufficient; it seems necessary to divide the theoretical real potential which was considered here by at least a factor 2.

4. – Conclusion.

In order to fit the experimental large total cross-sections and small elastic cross-sections, it is necessary to utilize a real potential of moderate strength, but a long range (at least around $\hbar/\mu c$) imaginary potential.

This last requirement could perhaps be met in a model [8] inspired by the fixed source meson theory, but improved so as to take also into account annihilation effects. Since the total cross-section is in first approximation proportional to the depth for an imaginary potential (instead of the square of the depth as for a real potential), a relatively small imaginary potential alone would be enough to produce a large total cross-section.

It is much more difficult to obtain a real potential which would be sufficiently small. The straightforward extension of the Gartenhaus potential to the nucleon-antinucleon system predicts a too large real potential. If additional annihilation-type graphs do not provide the destructive interference needed for obtaining a smaller real potential, it will appear difficult to keep the spirit of the fixed source meson theory for the nucleon-antinucleon interaction, even in the relatively low energy domain which was considered here. The necessity of a resort to the relativistic theory seems more and more difficult to discard.

* * *

We gratefully acknowledge Professor M. Lévy for having proposed this problem, and for his continuous interest. We thank Mr. R. A. Bryan for having kindly made available to us tabulations of the Gartenhaus potential.

The numerical computations were performed by courtesy of the IBM-France Company, through their European Research Fund, and we are especially indebted to Miss. F. RATTAUD for her help in programming.

APPENDIX I

The elastic cross-section is obtained from the first-order scattering amplitude $f^{(1)}(\theta)$:

$$\sigma_{\text{el}} = \int |f^{(1)}(\theta)|^2 \, \mathrm{d}\Omega = 4\pi M^2 \left[\frac{V_R^2}{\alpha^4(\alpha^2+4k^2)} + \frac{V_I^2}{\beta^4(\beta^2+4k^2)}\right], \tag{1}$$

where k is the initial (and final) momentum magnitude, and $M/2$ the reduced mass. We put $\hbar = c = \mu = 1$.

The total cross-section can be evaluated using the optical theorem. Since the elastic cross-section is of second order in the potential, consistency requires that the forward scattering amplitude be evaluated up to the second Born approximation:

$$\sigma_{\text{tot}} = \sigma_{\text{tot}}^{(1)} + \sigma_{\text{tot}}^{(2)}, \tag{2}$$

where:

$$\sigma_{\text{tot}}^{(1)} = 4\pi M V_I/(k\beta^3), \tag{3}$$

$$\sigma_{\text{tot}}^{(2)} = 4\pi M^2 \left[\frac{V_R^2}{\alpha^4(\alpha^2+4k^2)} - \frac{V_I^2}{\beta^4(\beta^2+4k^2)} + \frac{V_R V_I}{k^2\alpha\beta(\alpha^2-\beta^2)} \operatorname{tg}^{-1}\frac{2k(\alpha-\beta)}{\alpha\beta+4k^2}\right]. \tag{4}$$

It is seen by inspection of formulae (1), (3) and (4) that the Born approximation converges (at least up to the second order) when the elastic cross-section is small compared to the total cross-section, which is precisely the situation here.

APPENDIX II

Let us first consider the case of a triplet state $L = J$. The radial wave function has two real components:

$$\varphi_J(kx) = u_J(kx) + i w_J(kx) .$$

The 2 coupled real differential equations obeyed by u_J and w_J admit 2 systems of linearly independent solutions $\varphi^{(1)}$ and $\varphi^{(2)}$, corresponding to different boundary

conditions at the core; if this boundary condition is fixed, $\varphi(kx)$ is determined and has the asymptotic form:

$$\varphi(kx) \sim \frac{\exp\,[i\delta_{JJ}]\,\sin\,(kx - J\pi/2 + \delta_{JJ})}{kx},$$

where δ_{JJ} is a complex phase-shift independent of m ($m = \pm 1$). The calculated solution is analysed, outside the range of the potential, in terms of spherical Bessel functions, with two complex constants A and B:

$$\varphi = Aj_J(kx) + Bn_J(kx)\,.$$

By identification we obtain the phase-shift in the useful form $\exp\,[2i\delta_{JJ}]$:

$$\exp\,[2i\delta_{JJ}] = \frac{A + iB}{A - iB}\,.$$

For the spherical Bessel functions, the following asymptotic forms are used:

$$j_J(kx) \sim \frac{\sin\,(kx - J\pi/2)}{kx}\,, \qquad n_J(kx) \sim \frac{\cos\,(kx - J\pi/2)}{kx}\,.$$

We now study the more complicated case of the triplet states of parity $(-1)^{J+1}$. The waves of orbital angular momenta $L = J \pm 1$ are coupled by the real tensor forces. The radial wave functions:

$$\varphi_J(kx) = u_J^{J+1}(kx) + iw_J^{J+1}(kx)\,,$$
$$\psi_J(kx) = u_J^{J-1}(kx) + iw_J^{J-1}(kx)\,,$$

obey a system of 4 coupled real differential equations, which are easily obtained from those used for a real potential ([12]). For a fixed boundary condition at the core, there are two linearly independent solutions. The general solution is of the form:

$$\varphi = C_1\varphi^{(1)} + C_2\varphi^{(2)}\,,$$
$$\psi = C_1\psi^{(1)} + C_2\psi^{(2)}\,,$$

where C_1 and C_2 are two complex constants.

The asymptotic form of φ and ψ, for a given m, should be

$$\varphi(kx) \sim D_{J\,J+1}^{(m)}\,\frac{\exp\,[i\delta^m_{J\,J+1}]\,\sin\,(kx - (J+1)\pi/2 + \delta^m_{J\,J+1})}{kx}\,,$$

$$\psi(kx) \sim D_{J\,J-1}^{(m)}\,\frac{\exp\,[i\delta^m_{J\,J-1}]\,\sin\,(kx - (J-1)\pi/2 + \delta^m_{J\,J-1})}{kx}\,.$$

([12]) M. GOURDIN: *Journ. Phys. Rad.*, **17**, 988 (1956).

The D are constants related to Clebsch-Gordan coefficients:

$$D^0_{J\,J+1} = \quad D^{\pm1}_{J\,J-1} = \sqrt{\frac{J+1}{2J+1}}\,,$$

$$D^0_{J\,J-1} = - D^{\pm1}_{J\,J+1} = \sqrt{\frac{J}{2J+1}}\,.$$

The last requirement determines the constants C_1 and C_2.

The two independent systems $\varphi^{(1)}$, $\psi^{(1)}$ and $\varphi^{(2)}$, $\psi^{(2)}$ are analysed, as in the previous case, in spherical Bessel functions:

$$\left|\begin{aligned} \varphi^{(1)} &= A^{(1)}_+ j_{J+1}(kx) + B^{(1)}_+ n_{J+1}(kx)\,,\\ \psi^{(1)} &= A^{(1)} j_{J-1}(kx) + B^{(1)}_- n_{J-1}(kx)\,,\end{aligned}\right.$$

$$\left|\begin{aligned} \varphi^{(2)} &= A^{(2)}_+ j_{J+1}(kx) + B^{(2)}_+ n_{J+1}(kx)\,,\\ \psi^{(2)} &= A^{(2)}_- j_{J-1}(kx) + B^{(2)}_- n_{J-1}(kx)\,.\end{aligned}\right.$$

We finally obtain 4 phase-shifts: $\delta^0_{J\,J+1}$, $\delta^{\pm1}_{J\,J+1}$, $\delta^0_{J\,J-1}$, $\delta^{\pm1}_{J\,J-1}$:

$$\exp\,[2i\delta^0_{J\,J+1}] = \frac{R + iS + 2i\sqrt{J/(J+1)}\,T}{D}\,,$$

$$\exp\,[2i\delta^0_{J\,J-1}] = \frac{R - iS - 2i\sqrt{J/(J+1)}\,U}{D}\,,$$

$$\exp\,[2i\delta^{\pm1}_{J\,J+1}] = \frac{R + iS - 2i\sqrt{(J+1)/J}\,T}{D}\,,$$

$$\exp\,[2i\delta^{\pm1}_{J\,J-1}] = \frac{R - iS + 2i\sqrt{J/(J+1)}\,U}{D}\,.$$

R, S, T, U, D are complex numbers, combinations of the A and the B:

$$\begin{aligned}
R &= (A^{(1)}_+ A^{(2)}_- - A^{(2)}_+ A^{(1)}_-) + (B^{(1)}_+ B^{(2)}_- - B^{(2)}_+ B^{(1)}_-)\,,\\
S &= (B^{(1)}_+ A^{(2)}_- - B^{(2)}_+ A^{(1)}_-) - (A^{(1)}_+ B^{(2)}_- - A^{(2)}_+ B^{(1)}_-)\,,\\
T &= A^{(1)}_+ B^{(2)}_+ - B^{(1)}_+ A^{(2)}_+\,,\\
U &= A^{(1)}_- B^{(2)}_- - B^{(1)}_- A^{(2)}_-\,,\\
D &= (A^{(1)}_+ - iB^{(1)}_+)(A^{(2)}_- - iB^{(2)}_-) - (A^{(1)}_- - iB^{(1)}_-)(A^{(2)}_+ - iB^{(2)}_+)\,.
\end{aligned}$$

There is a relation between T and U, which is of the Wronskian type:

$$T + U = 0\,.$$

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It is easy to see that the phase-shifts are not independent and satisfy some relations analogous to those given by Schwinger in the case of a real potential:

$$\exp\,[2i\delta^{\pm 1}_{J\,J+1}] - \exp\,[2i\delta^{0}_{J\,J+1}] = \exp\,[2i\delta^{\pm 1}_{J\,J-1}] - \exp\,[2i\delta^{0}_{J\,J-1}]\,.$$

It is convenient to use a matrix notation for the numerical computation of the wave function.

The cross-sections are obtained from the phase-shifts by the usual formulae.

Note added in proof.

According to very recent and preliminary experimental results (private communication from O. CHAMBERLAIN), there are indications for some scattering at 150 MeV; the elastic cross-section would lie between 40 and 80 mb. As it has been pointed out in the above paper, it is evident that the larger is the elastic cross-section, the less important is the needed imaginary potential. If the elastic cross-section happens to be of an order as large as 80 mb, the model of Ball and Chew would nicely explain the results, as it can be seen from Table II.

RIASSUNTO (*)

Le sezioni efficaci nucleone-antinucleone sono state calcolate a 167 MeV con due modelli a buca di potenziale complessa. Si sono prima studiate buche complesse di Yukawa in approssimazione di Born. Si è successivamente analizzato sull'ordinatore elettronico IBM 704 un potenziale semiteorico: la parte reale è il potenziale di Signel e Marshak adattato al problema nucleone-antinucleone, e la parte immaginaria una buca fenomenologica di Yukawa. Per render conto dei risultati sperimentali (grande sezione efficace totale e piccola sezione efficace di diffusione), risulta necessario che il potenziale reale sia debole e che il potenziale immaginario abbia un range almeno uguale a quello usuale delle forze nucleari. La teoria mesonica della sorgente fissa da un lato conduce a un potenziale reale troppo forte e d'altra parte non predice alcun potenziale immaginario; questa teoria non può render conto dei risultati sperimentali.

(*) *Traduzione a cura della Redazione.*

On the Polarization of Electrons in μ-Meson Decay.

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(ricevuto l'11 Febbraio 1958)

In the recent paper by H. Überall ([1]) the polarization of electrons emitted in μ-meson decay has been calculated assuming the two-component theory for neutrinos.

We should like to present in this note the result which follows from the Fermi interaction described by the Lagrangian ([2])

$$(1)\qquad L' = \sum_{k=1}^{5} (\overline{\Psi}_e \Gamma_k \Psi_\mu)\big\{(\overline{\Psi}_\nu \Gamma_k (g_k^{(1)} + i g_k^{(1)'} \gamma_5)\Psi_\nu) + (\overline{\Psi}'_\nu \Gamma_k (g_k^{(2)} + i g_k^{(2)'} k_5)\Psi_\nu) + \\ + (\overline{\Psi}_\nu \Gamma_k (g_k^{(3)} + i g_k^{(3)'} \gamma_5)\Psi'_\nu)\big\} + \text{h.c.}\,,$$

where $\Psi'_\nu = C\overline{\Psi}_\nu$, $\overline{\Psi}_\nu = -\Psi_\nu C^{-1}$, $C^{-1}\gamma^\mu C = -\gamma^{\mu T}$, $\Gamma_1 = 1$, $\Gamma_2 = \gamma^\mu$, $\Gamma_3 = (1/2\sqrt{2})\cdot(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$, $\Gamma_4 = \gamma^\mu\gamma_5$, $\Gamma_5 = \gamma_5 = \gamma^1\gamma^2\gamma^3\gamma^4$, and $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}$, $g^{44} = -1$. The couplings proportional to the coupling constants $g_2^{(k)}$, $g_3^{(k)}$, $g_3^{(k)'}$, $g_4^{(k)'}$, $k = 2, 3$, vanish identically. For real coupling constants such Lagrangian will be invariant under time inversion. In the following the mass of the neutrino is assumed to be zero.

If we denote the direction of polarization of the meson at rest by $\boldsymbol{\lambda}$, $|\boldsymbol{\lambda}| = \lambda$ being the degree of polarization, the spin of the electron at rest by $\boldsymbol{e}$, and the direction of emission of the electron by $\boldsymbol{n} = \boldsymbol{p}/p$, the transition probability in unit time for a μ-meson at rest will be of the form

$$(2)\qquad \mathrm{d}w = \frac{1}{(2\pi)^4}\left(\frac{m_\mu}{2}\right)^5 F(\boldsymbol{\lambda}, \boldsymbol{e}, \boldsymbol{n}, x)\sqrt{1-\left(\frac{2\mu}{x}\right)^2}\, x^2\,\mathrm{d}x\,\mathrm{d}\Omega\,.$$

Here $x = 2E/m_\mu$, $E = \sqrt{p^2 + m_e^2}$, $\mu = m_e/m_\mu$, and $F(\boldsymbol{\lambda}, \boldsymbol{e}, \boldsymbol{n}, x)$ is equal to

$$(3)\qquad F(\boldsymbol{\lambda}, \boldsymbol{e}, \boldsymbol{n}, x) = f_1(x) + (\boldsymbol{e}\cdot\boldsymbol{\lambda})f_2(x) + (\boldsymbol{e}\cdot\boldsymbol{n})(\boldsymbol{\lambda}\cdot\boldsymbol{n})f_3(x) \pm \\ \pm (\boldsymbol{e}\cdot\boldsymbol{n})f_4(x) \pm (\boldsymbol{\lambda}\cdot\boldsymbol{n})f_5(x) + (\boldsymbol{e}\cdot\boldsymbol{\lambda}\times\boldsymbol{n})f_6(x)\,,$$

([1]) H. Überall: *Nuovo Cimento*, **6**, 376 (1957).

([2]) After this paper was finished the work by Kinoshita and Sirlin on the same problem, published in *Phys. Rev.*, **108**, 844 (1957), came to our notice. Our results agree with theirs. Our formula (7), for $\mu/x \ll 1$ and $\lambda = 1$, is identical with their formula (3.1). Prof. Dallaporta kindly informed us that this problem was also considered by Gatto and Lüders.

where the upper (lower) sign is for particles (antiparticles). For the Lagrangian (1) the functions $f_i(x)$ are as follows

$$
(4)\quad \left\{
\begin{aligned}
f_1(x) &= \frac{1}{2}\alpha_1(1-x+\mu^2)\left(1+\frac{2\mu}{x}\right)+\alpha_2\left(1-\frac{2}{3}x+\frac{2}{3}\mu+\mu^2\right)\left(1-\frac{2\mu}{x}\right)+\\
&\quad+\alpha_3\left(1-\frac{1}{3}x-\frac{8}{3}\frac{\mu^2}{x}+\mu^2\right)+\alpha_4\left(1-\frac{2}{3}x-\frac{2}{3}\mu+\mu^2\right)\left(1+\frac{2\mu}{x}\right)+\\
&\quad+\frac{1}{2}\alpha_5(1-x+\mu^2)\left(1-\frac{2\mu}{x}\right),\\
f_2(x) &= \frac{1}{2}\alpha_1(1-x+\mu^2)\left(1+\frac{2\mu}{x}\right)+\frac{1}{3}\alpha_2(1+\mu)^2\left(1-\frac{2\mu}{x}\right)-\\
&\quad-\frac{2}{3}\alpha_3\frac{\mu}{x}(1-x+\mu^2)-\frac{1}{3}\alpha_4(1-\mu)^2\left(1+\frac{2\mu}{x}\right)-\\
&\quad-\frac{1}{2}\alpha_5(1-x+\mu^2)\left(1-\frac{2\mu}{x}\right),\\
f_3(x) &= -\frac{2}{3}\alpha_2(1-x+\mu^2)\left(1-\frac{2\mu}{x}\right)-\frac{1}{3}\alpha_3(1+x+4\mu+\mu^2)\left(1-\frac{2\mu}{x}\right)+\\
&\quad+\frac{2}{3}\alpha_4 x\left[1-\left(\frac{2\mu}{x}\right)^2\right]+\alpha_5(1-x+\mu^2)\left(1-\frac{2\mu}{x}\right),\\
f_4(x) &= -\alpha_6(1-x+\mu^2)\sqrt{1-\left(\frac{2\mu}{x}\right)^2}+2\alpha_8\left(1-\frac{2}{3}x+\frac{1}{3}\mu^2\right)\sqrt{1-\left(\frac{2\mu}{x}\right)^2}+\\
&\quad+\alpha_{10}\left(1-\frac{1}{3}x-\frac{1}{3}\mu^2\right)\sqrt{1-\left(\frac{2\mu}{x}\right)^2},\\
f_5(x) &= -\alpha_6(1-x+\mu^2)\sqrt{1-\left(\frac{2\mu}{x}\right)^2}-\frac{2}{3}\alpha_8(1-2x+3\mu^2)\sqrt{1-\left(\frac{2\mu}{x}\right)^2}-\\
&\quad-\frac{1}{3}\alpha_{10}(1+x-3\mu^2)\sqrt{1-\left(\frac{2\mu}{x}\right)^2},\\
f_6(x) &= \alpha_7(1-x+\mu^2)\sqrt{1-\left(\frac{2\mu}{x}\right)^2}-\frac{2}{3}\alpha_9(1-\mu^2)\sqrt{1-\left(\frac{2\mu}{x}\right)^2}.
\end{aligned}\right.
$$

The then parameters α_i are [3]

$$
\alpha_i = \alpha_i^{(1)} + 2(\alpha_i^{(2)} + \alpha_i^{(3)})\,, \qquad i = 1, \ldots, 10,
$$

---

(2) The combinations of coupling constants which appear here are invariant under the transformation considered by PAULI: *Nuovo Cimento*, **6**, 204 (1957).

where

$$\alpha_i^{(k)} = |g_i^{(k)}|^2 + |g_i^{(k)'}|^2 , \qquad i = 1, \dots, 5,$$

$$\alpha_6^{(k)} = \mathrm{Re}\,\beta_{15}^{(k)}, \quad \alpha_7^{(k)} = \mathrm{Im}\,\beta_{15}^{(k)}, \quad \alpha_8^{(k)} = \mathrm{Re}\,\beta_{24}^{(k)}, \quad \alpha_9^{(k)} = \mathrm{Im}\,\beta_{24}^{(k)}, \quad \alpha_{10}^{(k)} = \beta_{33}^{(k)},$$

$$\beta_{ij}^{(k)} = g_i^{(k)} g_j^{(k)'*} + g_i^{(k)'} g_j^{(k)*},$$

$$g_2^{(k)} = g_3^{(k)} = g_3^{(k)'} = g_4^{(k)'} = 0 , \qquad k = 2, 3 .$$

The polarization effects are given by the function F, which can also be written in the form:

$$(5) \qquad \begin{cases} F = a + \boldsymbol{b}\cdot\boldsymbol{e} , \\ a = f_1(x) \pm (\boldsymbol{\lambda}\cdot\boldsymbol{n}) f_5(x) , \\ \boldsymbol{b} = b_1 \dfrac{(\boldsymbol{n}\times\boldsymbol{\lambda})\times\boldsymbol{n}}{|\boldsymbol{n}\times\boldsymbol{\lambda}|} + b_2 \dfrac{\boldsymbol{n}\times\boldsymbol{\lambda}}{|\boldsymbol{n}\times\boldsymbol{\lambda}|} + b_3\boldsymbol{n} , \\ b_1 = |\boldsymbol{n}\times\boldsymbol{\lambda}| f_2(x) , \\ b_2 = -|\boldsymbol{n}\times\boldsymbol{\lambda}| f_6(x) , \\ b_3 = (\boldsymbol{\lambda}\cdot\boldsymbol{n})(f_2(x) + f_3(x)) \pm f_4(x) . \end{cases}$$

From this expression we see that the direction and the degree of polarization are given by $\boldsymbol{b}$ and $|\boldsymbol{b}|/a$, respectively. If the Lagrangian (1) is invariant under time inversion, $f_6 = 0$, so that $b_2 = 0$, i.e. the electron will be polarized in the scattering plane $(\boldsymbol{\lambda}, \boldsymbol{n})$. The Michel parameter is given by

$$(6) \qquad \varrho = \frac{3(\alpha_2 + \alpha_4) + 6\alpha_3}{\alpha_1 + \alpha_5 + 4(\alpha_2 + \alpha_4) + 6\alpha_3} .$$

If we denote the angle between $\boldsymbol{\lambda}$ and $\boldsymbol{n}$ by θ and neglect terms of higher order in μ, but not in μ/x, we obtain from (4) and (5)

$$(7) \qquad \begin{cases} a = \dfrac{1}{2}(\alpha_1 + \alpha_5)(1 - x) + (\alpha_2 + \alpha_4)\left(1 - \dfrac{2}{3}x\right) + \alpha_3\left(1 - \dfrac{1}{3}x\right) + \\ \qquad + [(\alpha_1 - \alpha_5) - 2(\alpha_2 - \alpha_4)]\dfrac{\mu}{x} \mp \\ \qquad \mp \lambda\cos\theta\left[\alpha_6(1 - x) + \dfrac{2}{3}\alpha_8(1 - 2x) + \dfrac{1}{3}\alpha_{10}(1 + x)\right]\sqrt{1 - \left(\dfrac{2\mu}{x}\right)^2} , \\ b_1 = \lambda\sin\theta\left[\dfrac{1}{2}(\alpha_1 - \alpha_5)(1 - x) + \dfrac{1}{3}(\alpha_2 - \alpha_4) + \left[(\alpha_1 + \alpha_5) - \right.\right. \\ \qquad \left.\left. - \dfrac{2}{3}(\alpha_2 + \alpha_4) - \dfrac{2}{3}\alpha_3\right]\dfrac{\mu}{x}\right] , \end{cases}$$

$$
(7)\quad \left\{
\begin{aligned}
b_2 &= \lambda \sin\theta \left[-\alpha_7(1-x)+\frac{2}{3}\alpha_9\right]\sqrt{1-\left(\frac{2\mu}{x}\right)^2}\,,\\
b_3 &= \lambda \cos\theta \left[\frac{1}{2}(\alpha_1+\alpha_5)(1-x)-\frac{1}{3}(\alpha_2+\alpha_4)(1-2x)-\frac{1}{3}\alpha_3(1+x)+\right.\\
&\quad + \left.\left[(\alpha_1-\alpha_5)+\frac{2}{3}(\alpha_2-\alpha_4)\right]\frac{\mu}{x}\right] \pm\\
&\quad \pm\left[-\alpha_6(1-x)+2\alpha_8\left(1-\frac{2}{3}x\right)+\alpha_{10}\left(1-\frac{1}{3}x\right)\right]\sqrt{1-\left(\frac{2\mu}{x}\right)^2}\,.
\end{aligned}
\right.
$$

The experimental knowledge of the polarization of electrons at all energies and angles, assuming the polarization of μ-meson to be known, would enable us to determine the parameters α_i. At present this is not possible since the experiments are far from being complete, particularly in the case of the electron polarization, where the situation is confusing. To decide whether the μ-meson decay interaction is more general than as it is given by the two-component theory, it will be necessary to perform new experiments, particularly on the polarization. Here we should like to point out that even if the two-component theory would eventually explain the data on μ-meson decay, this would not be, strictly speaking, the proof of its validity. This is due to the fact that the coupling constants $g_i^{(k)}$, $g_i^{(k)'}$ are not uniquely determined by the α_i's.

Two Components Spinor Equation.

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(ricevuto il 28 Febbraio 1958)

We know that the general solution of the neutrino equation

$$\gamma_\mu \partial_\mu \psi = 0 , \tag{1}$$

may be written:

$$\psi = \psi^{(1)} + \psi^{(2)} , \tag{2}$$

where:

$$\psi^{(1)} = \frac{(1+\gamma_5)}{2}\,\psi , \qquad \psi^{(2)} = \frac{(1-\gamma_5)}{2}\,\psi , \tag{3}$$

satisfy

$$\begin{cases} \gamma_\mu \partial_\mu \psi^{(1)} = 0 , \\ (1-\gamma_5)\psi^{(1)} = 0 , \end{cases} \qquad \begin{cases} \gamma_\mu \partial_\mu \psi^{(2)} = 0 , \\ (1+\gamma_5)\psi^{(2)} = 0 . \end{cases} \tag{4}$$

Moreover, with the usual representation of the γ_μ's, we may put:

$$\psi^{(1)} = \begin{pmatrix} \varphi^{(1)} \\ -\varphi^{(1)} \end{pmatrix} , \qquad \psi^{(2)} = \begin{pmatrix} \varphi^{(2)} \\ \varphi^{(2)} \end{pmatrix} , \tag{5}$$

where $\varphi^{(1)}$ and $\varphi^{(2)}$ are two-components spinors satisfying:

$$(\boldsymbol{\sigma}\cdot\boldsymbol{\partial} - i\partial_4)\varphi^{(1)} = 0 , \qquad (\boldsymbol{\sigma}\cdot\boldsymbol{\partial} + i\partial_4)\varphi^{(2)} = 0 . \tag{6}$$

The possibility of separating the neutrino equation into two-components equa-

tions may be also expressed by saying that, if $\psi^{(a)}$ and $\psi^{(b)}$ are arbitrary solutions,

$$\psi' = (1+\gamma_5)\psi^{(a)} + (1-\gamma_5)\psi^{(b)} \tag{7}$$

is also a solution.

Let us now search if it is possible to write a spinor equation of the kind of that introduced by HEISENBERG ([1]), having the above mentioned property (formula (7)) of the neutrino equation.

To obtain this we write an interaction Lagrangian which is the sum of two terms, one containing only $(1+\gamma_5)\psi$ and its conjugate $\bar{\psi}(1-\gamma_5)$, the other containing in the same form only $(1-\gamma_5)\psi$ and its conjugate $\bar{\psi}(1+\gamma_5)$ ([2]). As it is clear from the analysis of Feynman and Gell-Mann, the only possible interaction Lagrangian having this property is ([3,4]).

$$L' = \frac{l^2}{2}\left(\bar{\psi}\gamma_\mu \frac{1+\gamma_5}{2}\psi\right)\left(\bar{\psi}\gamma_\mu \frac{1+\gamma_5}{2}\psi\right) + \left(\bar{\psi}\gamma_\mu \frac{1-\gamma_5}{2}\psi\right)\left(\bar{\psi}\gamma_\mu \frac{1-\gamma_5}{2}\psi\right), \tag{8}$$

so that the corresponding equation is:

$$\gamma_\mu \partial_\mu \psi + l^2\left(\gamma_\mu \frac{1+\gamma_5}{2}\psi\left(\bar{\psi}\gamma_\mu \frac{1+\gamma_5}{2}\psi\right) + \gamma_\mu \frac{1-\gamma_5}{2}\psi\left(\bar{\psi}\gamma_\mu \frac{1-\gamma_5}{2}\psi\right)\right) = 0\,. \tag{9}$$

It is clear that this equation has the property analogous to the one expressed by the formula (7), and therefore it may be reduced to two-components equations, analogously to the neutrino equation. With the positions (2) and (3) we obtain, instead of (4):

$$\left\{\begin{aligned} &\gamma_\mu \partial_\mu \psi^{(1)} + l^2 \gamma_\mu \psi^{(1)}(\bar{\psi}^{(1)}\gamma_\mu \psi^{(1)}) = 0\,, \\ &(1-\gamma_5)\psi^{(1)} = 0\,, \end{aligned}\right. \tag{10$'$}$$

$$\left\{\begin{aligned} &\gamma_\mu \partial_\mu \psi^{(2)} + l^2 \gamma_\mu \psi^{(2)}(\bar{\psi}^{(2)}\gamma_\mu \psi^{(2)}) = 0\,, \\ &(1+\gamma_5)\psi^{(2)} = 0\,. \end{aligned}\right. \tag{10$''$}$$

With the positions (5) the equation (10′) becomes ([5]):

$$i(\boldsymbol{\sigma}\cdot\boldsymbol{\partial} - i\partial_4)\varphi - 2l^2(\boldsymbol{\sigma}\varphi\cdot(\varphi^*\boldsymbol{\sigma}\varphi) - \varphi(\varphi^*\varphi)) = 0\,, \tag{11}$$

([1]) For the bibliography on this subject see: W. HEISENBERG: *Rev. Mod. Phys.*, **29**, 269 (1957).

([2]) The interest of interactions containing only the quantities $(1+\gamma_5)\psi$ or $(1-\gamma_5)\psi$ has been specially pointed out by R. P. FEYNMAN and M. GELL-MANN: *A theory of the Fermi interaction* (preprint).

([3]) The Lagrangian (8) or the right hand side of equation (9) have the property of being identically equal to zero, if the field is not quantized, with anticommutators. This point will be discussed particularly in the Appendix.

([4]) The presence of scalar or pseudoscalar mass terms of the kind $m\bar{\psi}\psi$ or $m\bar{\psi}\gamma_5\psi$ is excluded by the same kind of arguments.

([5]) The equation derived from (10″) differs only for the sign before $\boldsymbol{\partial}$.

or explicitly with

$$\varphi = \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}, \tag{12}$$

$$\begin{cases} i(\partial_1 - i\,\partial_2)\varphi_2 + i(\partial_3 - \partial_0)\varphi_1 + 4l^2\varphi_1[\varphi_2^*\varphi_2] = 0\,, \\ i(\partial_1 + i\,\partial_2)\varphi_1 + i(-\partial_3 - \partial_0)\varphi_2 + 4l^2\varphi_2[\varphi_1^*\varphi_1] = 0\,. \end{cases} \tag{13}$$

Correspondingly the part of the interaction Lagrangian (8) containing $\varphi^{(1)}$ becomes:

$$-l^2(\varphi^*\boldsymbol{\sigma}\varphi)(\varphi^*\boldsymbol{\sigma}\varphi) - (\varphi^*\varphi)(\varphi^*\varphi)) = 2l^2(\varphi_1^*\varphi_1[\varphi_2^*\varphi_2] + \varphi_2^*\varphi_2[\varphi_1^*\varphi_1])\,. \tag{14}$$

Therefore it seems possible to write a non linear two components spinor equation, provided the spinor field is quantized with anticommutators. It is easy to see that the (10′) or (11) or (13) is the only possible non linear two components spinor equation for quantized fields. In fact there are only four independent field variables, φ_1, φ_2, φ_1^*, φ_2^*. After quantization the square of any one of them is 0, owing to $\{\varphi_1\varphi_1\} = 0$ and so on. Now any interaction term more complicated than (8) will contain products of more than four field variables; therefore it shall either vanish or reduce itself to terms with no more than four field variables [6,7]. Therefore, if the equation (10) or (11) or (13) has a meaning, it is the only possible two components non linear spinor equation [8].

The Lagrangian (8) or (14) is invariant under the transformations:

$$\psi' = \exp[i\alpha]\psi\,, \qquad \psi' = \exp[i\alpha\gamma_5]\psi\,, \qquad \psi' = b\gamma_5\psi^c\,, \qquad (|b|^2 = 1)\,, \tag{15}$$

but it is not invariant under the Pauli transformation

$$\psi' = a\psi + b\gamma_5\psi^c\,, \qquad (|a|^2 + |b|^2 = 1)\,, \tag{16}$$

for $a \neq 0$ and $b \neq 0$.

Therefore it has not all the invariance properties of the classical interaction Lagrangian

$$\left(\bar\psi\,\frac{1-\gamma_5}{2}\,\psi\right)\left(\bar\psi\,\frac{1+\gamma_5}{2}\,\psi\right), \tag{17}$$

that has been lately proposed by HEISENBERG as a basis for a theory of all elementary particles; however it has, instead of the property (16), the property

---

(6) Considerations of this kind have been used by W. E. THIRRING, in connection with his two dimensional spinor model: W. E. THIRRING: *A soluble relativistic field theory* (preprint).

(7) Using a rather crude definition, let us call elementary particle a system the constituents of which may interact all at the same point. Then it follows from analogous considerations that elementary particles described by this spinor equation cannot have a spin greater than 1.

(8) The only existing ambiguity is the sign before l^2 which is matter of convention.

expressed by formula (7) or, what is the same, it is reducible to a two-components equation [9,10].

For this reasons the equation (9) or (10) or (11) or (13) may be of interest for a theory of the elementary particles.

* * *

The author thanks Professor HEISENBERG for having made available his results before publication, and Professor WATAGHIN for his kind interest to this work.

APPENDIX

The Lagrangian (8) or (14) has the property of being identically zero if the field is not quantized, and it has no classical analogue. Let us verify directly the invariance of the Lagrangian (14) for proper Lorentz transformations in the simpler case in which the quantization is done with the Heisenberg method [1]. Therefore we assume that all the field variables at a given time anticommute with each other and with themselves. Then (14) becomes:

$$L' = 8l^2\varphi_1^*\varphi_1\varphi_2^*\varphi_2\,. \tag{A.1}$$

The proper Lorentz transformations give rise to transformations for the φ of the form

$$\varphi' = \Sigma\varphi\,, \tag{A.2}$$

where Σ is a 2×2 matrix of determinant 1 [11]. Let us put:

$$\Sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad ad - bc = 1\,, \qquad \begin{cases} \varphi_1' = a\varphi_1 + b\varphi_2\,, \\ \varphi_2' = c\varphi_1 + d\varphi_2\,. \end{cases} \tag{A.3}$$

---

(9) Such an equation is simpler to handle than a four components equation, owing to the simpler algebra of the σ_i matrices compared to the γ_μ matrices.

(10) Another interesting spinor equation is obtained from the interaction Lagrangian (see note added in proof):

$$L' = \frac{l^2}{2}(\bar\psi\gamma_\mu\psi)(\bar\psi\gamma_\mu\psi) = \frac{l^2}{2}\left(\bar\psi\gamma_\mu\frac{1+\gamma_5}{2}\psi + \bar\psi\gamma_\mu\frac{1-\gamma_5}{2}\psi\right)\left(\bar\psi\gamma_\mu\frac{1+\gamma_5}{2}\psi + \bar\psi\gamma_\mu\frac{1-\gamma_5}{2}\psi\right).$$

Such an equation has all the invariance properties (15), (16), but it is not reducible to a two components equation. However it has the property of having solutions of the form

$$\psi^{(1)} = \begin{pmatrix} \varphi^{(1)} \\ -\varphi^{(1)} \end{pmatrix}, \qquad \psi^{(2)} = \begin{pmatrix} \varphi^{(2)} \\ \varphi^{(2)} \end{pmatrix},$$

[these solutions are respectively the solutions of the systems (10′), (10″) and are therefore obtained from the two components equation (11); but the sum of two solutions of this kind is no more a solution of the equation, as in the case of equation (9)].

(11) See for instance: W. PAULI: *Handb. d. Phys.*, 1, 148 (1958).

Then we have

$$\varphi_1'^*\varphi_1'\varphi_2'^*\varphi_2' = (a^*\varphi_1^* + b^*\varphi_2^*)(a\varphi_1 + b\varphi_2)(c^*\varphi_1^* + d^*\varphi_2^*)(c\varphi_1 + d\varphi_2)\,. \tag{A.4}$$

The only terms different from zero are those which do not contain twice the same field variable, owing to the anticommutativity. Therefore we have:

$$\begin{aligned}\varphi_1'^*\varphi_1'\varphi_2'^*\varphi_2' &= a^*ad^*d\varphi_1^*\varphi_1\varphi_2^*\varphi_2 + a^*bd^*c\varphi_1^*\varphi_2\varphi_2^*\varphi_1 + b^*ac^*d\varphi_2^*\varphi_1\varphi_1^*\varphi_2 + \\ &+ b^*bc^*c\varphi_2^*\varphi_2\varphi_1^*\varphi_1 = \varphi_1^*\varphi_1\varphi_2^*\varphi_2(a^*ad^*d - a^*bd^*c - b^*ac^*d + b^*bc^*c) = \\ &= \varphi_1^*\varphi_1\varphi_2^*\varphi_2\big(a^*d^*(ad-bc) - b^*c^*(ad-bc)\big) = \varphi_1^*\varphi_1\varphi_2^*\varphi_2\,.\end{aligned} \tag{A.5}$$

Therefore the quantity $\varphi_1^*\varphi_1\varphi_2^*\varphi_2$ is invariant under Lorentz transformations, when anticommuting algebra for the φ's, φ^*'s is used. It is immediately seen that it is not invariant if the φ's, φ^*'s are commuting quantities. In fact the right hand side of (A.4) contains for instance a term $a^*ac^*c\varphi_1^*\varphi_1\varphi_1^*\varphi_1$ which has certainly not the form $\varphi_1^*\varphi_1\varphi_2^*\varphi_2$.

Note added in proof.

Meanwhile a preprint with the names of W. HEISENBERG and W. PAULI has appeared. Their work confirms the possibility of choosing the interaction Lagrangian (8) only in a theory quantized with anticommutators: indeed it is $L' = \frac{1}{4}l^2(I_2 - I_4)$, where I_2 and I_4 are defined by formula (7) and satisfy (8) or (9) of their work. Moreover the comparison with their work has led us to find an error of sign in the derivation of the Lagrangian of footnote (10), so that the right formula is:

$$L' = \frac{l^2}{2}\left(\bar\psi\gamma_\mu\frac{1+\gamma_5}{2}\psi - \bar\psi\gamma_\mu\frac{1-\gamma_5}{2}\psi\right)\left(\bar\psi\gamma_\mu\frac{1+\gamma_5}{2}\psi - \bar\psi\gamma_\mu\frac{1-\gamma_5}{2}\psi\right) = \frac{l^2}{2}(\bar\psi\gamma_\mu\gamma_5\psi)(\bar\psi\gamma_\mu\gamma_5\psi)\,,$$

in agreement with the interaction Lagrangian proposed by HEISENBERG and PAULI.

On the Possibility of the Charge Dependent Correction to the Gravitational Forces (*).

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(ricevuto il 5 Marzo 1958)

Attempts have been made to explain the gravitational force in terms of the two neutrino exchange between two nucleons ([1,2]). Corben ([3]) compared the gravitational force with the force due to the two neutrino exchange by considering fourth order perturbation theory with respect to the actual Fermi interactions.

In this note we do not try to explain the entire gravitational force in terms of a combination of the Fermi interaction, but we emphasize that the force due to the Fermi interaction should be discussed to determine if it is important as a small correction to the gravitational force. As is well known, the accuracy of the experiments on gravitational phenomena is extremely high, so that it is certainly desirable to check any small correction.

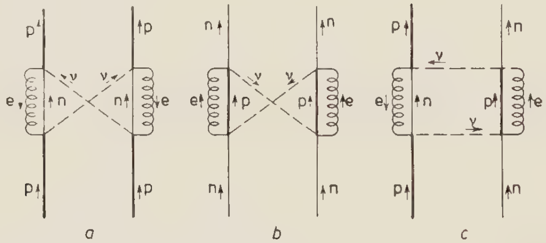

Fig. 1. – Fourth order Feynman diagrams for the long range nuclear forces.

First of all, we would like to discuss what would be expected if such a correction were important. The possible Feynman diagrams for p-p, n-n, and p-n forces are given in Figs. 1*a*, 1*b*, and 1*c*, respectively. One can see that p-p and n-n forces are somewhat different from the p-n forces, if the lepton number is conserved. This could be detected by astrophysical and geophys-

(*) Assisted by contract with the Air Force Office of Scientific Research.

(1) G. Gamow and E. Teller: *Phys. Rev.*, **51**, 289 (1937).

(2) G. Gamow: *Phys. Rev.*, **71**, 550 (1947).

(3) H. C. Corben: *Nuovo Cimento*, **10**, 1485 (1953).

ical measurements, if such a charge dependent correction were sufficiently large.

Fortunately, we can find the r dependence of this potential in the static limit, although it contains unrenormalizable divergences which we do not try to remove here. Applying the method which was used to get the fourth order nuclear potential, we obtain our potential with a leading term proportional to $1/r^3$, contrary to the $1/r$ dependence previously expected (3). Actually the potential V for the scalar Fermi coupling is:

$$V=\left(\frac{g}{\hbar c}\right)^4\frac{1}{\pi^3}\frac{1}{r^3}\times(\text{infinite quantity with dimension cm}^{-3})^2, \tag{1}$$

where g is the Fermi coupling constant (10^{-49} erg cm^3). For other Fermi couplings the r dependence is essentially the same. It turns out, therefore, that this correction is quite small.

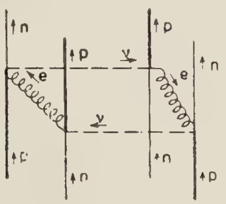

Fig. 2. – Many body force diagram which gives long range force. Here a pair of proton and neutron which are bound by an electron are supposed to be in the same nucleus.

Another example of a charge dependent correction is shown in Fig. 2. This correction occurs only for the interaction between pairs of p-n systems. The potential is again more singular than $1/r$.

A higher singularity is also present for the simplest potential from weak coupling (shown in Fig. 3), which is an

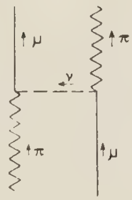

Fig. 3. – The simplest diagram for the weak long range force.

exchange force between pion and muon. In the latter case the potential is $1/r^2$, namely:

$$V=\frac{g^2}{4\pi\hbar^2c^2}\frac{\varrho_2(\boldsymbol{\sigma}\cdot\boldsymbol{r})}{r^3}, \tag{2}$$

where $g^2/4\pi\hbar c$ is the dimensionless coupling constant of $(\mu\nu\pi)$. ϱ and σ are Dirac matrices.

Our problem is quite different from the derivation of the Coulomb potential in the two neutrino theory of light (4), where it was necessary to introduce a combination different from the Fermi interactions. Not discussed here are some questions on the r dependence which still remain if one introduces an artificial technique to remove infinities in the integral (5,6).

It is concluded that the Fermi interaction does not produce any detectable correction to the gravitational force.

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(4) A. Sokolow: *Physica*, **5**, 797 (1938).

(5) P. Bocchieri and P. Gulmanelli: *Nuovo Cimento*, **5**, 1016 (1957).

(6) P. Gulmanelli and E. Montaldi: *Nuovo Cimento*, **5**, 1716 (1957).

IL NUOVO CIMENTO VOL. VIII, N. 3 1° Maggio 1958

Angular Correlation of Annihilation Radiation in Water, Ice and Mercury.

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(Ricevuto il 20 Marzo 1958)

The angular correlation of the γ-rays from positron annihilation is being investigated with an apparatus similar to that of LANG and DE BENEDETTI (1). In our instrument (Fig. 1) the annihilation source and the counter slits are horizontal, in order to make possible the study of the annihilation from the liquid surfaces. The liquids are contained in a small open cell 11.5 cm $\times$ 1 cm, and bombarded with positrons from a source (1 mC of ^{22}Na) situated a few mm above their surface.

The coincidence circuit used has a resolving time of the order of 10^{-8} s.

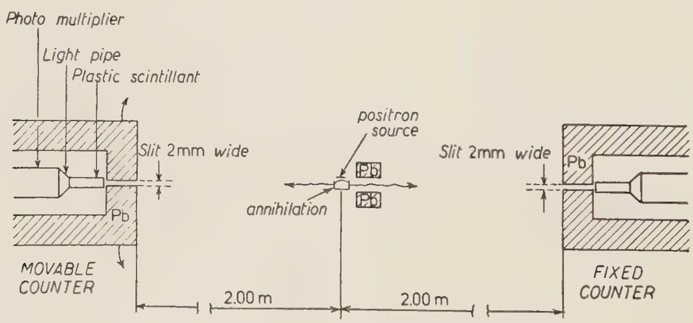

Fig. 1. – Vertical section of the apparatus (schematic). The distance between the source assembly and the detectors is not represented in scale.

As a first problem it was considered of some interest to investigate the angular correlation in water and ice (at 20 °C and — 30 °C respectively), in order to study the effect of the change of state. It is known that the behaviour of positrons may be affected by melting: in the case of naphthalene the long

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(1) G. LANG and S. DE BENEDETTI: *Phys. Rev.*, **108**, 914 (1957).

life τ_2 increases when one passes from the solid to the liquid phase, indicating a larger percentage of positronium in this work we were informed that a similar result had been obtained at the University of Maryland [6].

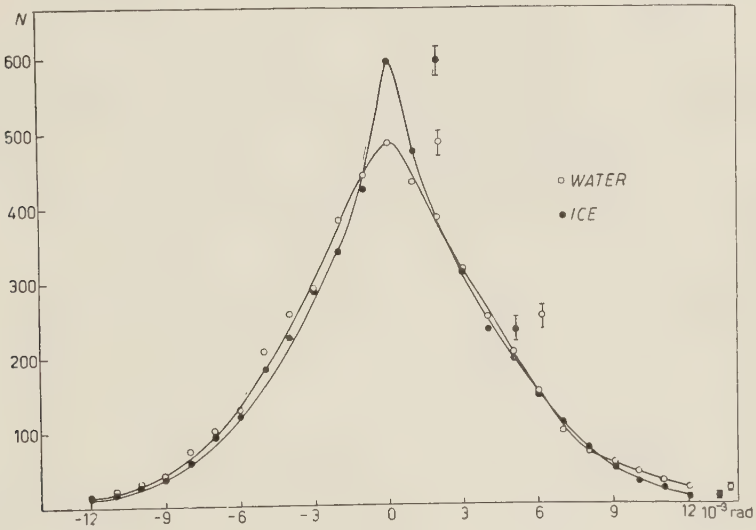

Fig. 2.

the liquid [2]. τ_2 is larger in water than in ice [3] though no abrupt change in the number of triplet positronium is observed at the melting point [4].

The experimental results obtained with water and ice are shown in Fig. 2. The central points have been repeated with an angular resolution of $0.5 \cdot 10^{-3}$ rad. The results are shown in Fig. 3. Contrary to our expectations the curve referring to ice shows a sharp peak, of the kind usually associated with positronium [5], while the liquid curve is more rounded. After the completion of

These observations are difficult to explain under the usual interpretation

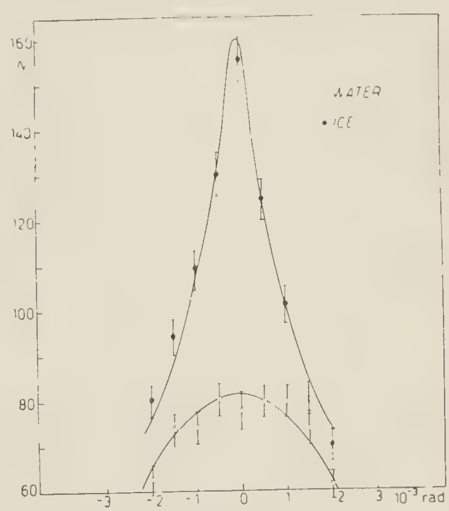

Fig. 3.

([2]) H. S. LANDER, S. BERKO and A. J. ZUCHELLI: *Phys. Rev.*, **103**, 828 (1956).

([3]) R. E. BELL and R. L. GRAHAM: *Phys. Rev.*, **90**, 644 (1953).

([4]) R. T. WAGNER and F. L. HEREFORD: *Phys. Rev.*, **99**, 593 (1955).

([5]) L. A. PAGE, M. HEINBERG, J. WALLACE and T. TROUT: *Phys. Rev.*, **98**, 206 (1955); A. T. STEWART: *Phys. Rev.*, **99**, 594 (1955).

([6]) R. L. DE ZAFRA and W. T. JOYNER: personal communication.

according to which both the peakedness of the angular distribution and the long mean life are attributable to thermal positronium.

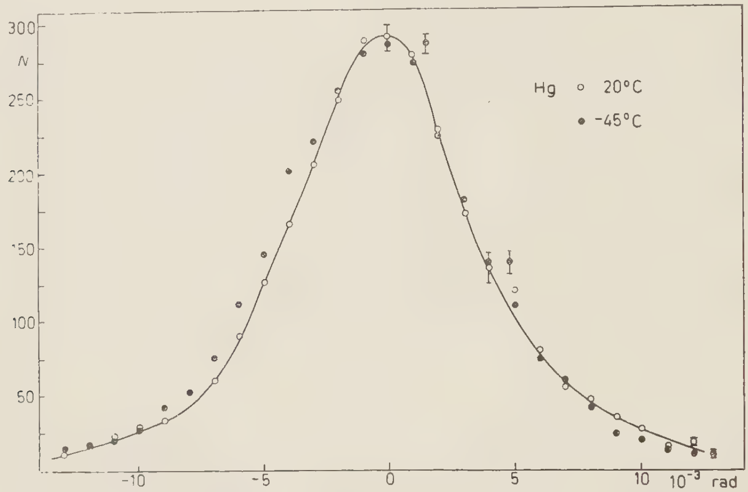

Fig. 4.

since in a metal the annihilation properties depend almost exclusively on the free electrons and these do not change with the temperature.

Measurements on liquid and solid Mercury (at 20 °C and — 45 °C respectively) gave the curve shown in Fig. 4.

This element does not show any significant difference between liquid and solid state. This was to be expected

Investigation is continuing on this problem and is extended to other substances.

* * *

We are grateful to Dr. L. MAJRONE for his very useful help in this work.

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Proposal for a Distributed Regenerative Action to Extract the Beam from a Weak-Focusing Synchrotron by Exciting the Resonance $\dfrac{2 \text{ Radial Oscillations}}{3 \text{ Revolutions}}$.

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(ricevuto il 29 Marzo 1958)

1. – Introduction.

Till now the external electron beam has not been achieved in high energy circular machines. Indeed, the regenerative [1-6] solution is disadvantaged by the fact that the required nominal value of the regenerator strength is not obtainable in practice because of;

i) the narrowness of the linear field region;

ii) the form of the radial restoring force.

The radial equation of motion may be written — neglecting terms of order x/R with respect to unity —

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\theta^2} + x + \frac{R}{B_0}(B_z - B_0) = 0\,, \qquad x = r - R, \tag{1.1}$$

or

$$\frac{\mathrm{d}^2 x}{\mathrm{d}\theta^2} + [1 - \bar{n}(x)]x = 0\,, \tag{1.2}$$

where

$$\bar{n}(x) = \frac{1}{x}\int_0^x n(x)\,\mathrm{d}x\,; \qquad n(x) = -\left[\frac{r}{B_z}\frac{\partial B_z}{\partial r}\right]_{z=0}. \tag{1.2'}$$

(1) A. V. Crewe and K. J. Le Couteur: *Rev. Sci. Instr.*, **26**, 725 (1955).
(2) A. V. Crewe and J. W. G. Gregory: *Proc. Roy. Soc. London*, A **232**, 242 (1955).
(3) A. V. Crewe and U. E. Kruse: *Rev. Sci. Instr.*, **27**, 5 (1956).
(4) J. L. Tuck and L. C. Teng: *Phys. Rev.*, **81**, 305 (1951).
(5) K. J. Le Couteur: *Proc. Phys. Soc.*, B **64**, 1073 (1951).
(6) S. Cohen and A. V. Crewe: *CERN Symposium on High Energy Accelerator* (1956), p. 140.

This means that it is very difficult to obtain an $(1/x)\int_0^x n(x)\,\mathrm{d}x$ required value having the following features: $\bar{n}_{\text{perturbed}} = \text{const} \gg 1$ only in a region of narrow angular width and only on one side of the equilibrium orbit. Equations (1.2), (1.2′) and their consequences are also valid for cyclotrons.

We shall now trace out the possibility of an azimuthally distributed regenerative action to extract the circulating electron beam from a racetrack having the operating point $[n_0, L/R]$ in the neighborhood of the resonance line (for betatron oscillations): 2 radial oscillations/3 revolutions. (This is, for example, the case of the 1 GeV Frascati Synchrotron: $\langle n_0 \rangle = 0.61$ designed field index value; four straight sections of lenght $L/R = 0.335$; resonance field index value $n_{\text{res}} = 0.635$, $R = 360$ cm).

For a right investigation of the betatron oscillations stability from this point of view, the racetrack must be replaced by a Circular Synchrotron having *the same periodicity* for radial oscillations.

For such a synchrotron the resonance $\langle n_0 \rangle$ value is $n_{\text{res}} = \frac{5}{9}$. The perturbing term most responsible for the radial oscillations growth is the non-linear term $\frac{1}{2}(\mathrm{d}n/\mathrm{d}x)^{(1,2)}x^2 \sin 2\theta$ derived from an $n(x, \theta)$ of the form

$$n(x, \theta) = \left(\frac{5}{9} - \delta\right) + \left(\frac{\mathrm{d}n}{\mathrm{d}x}\right)^{(1,2)} x \sin 2\theta\,, \tag{1.3}$$

where

$$\delta = n_{\text{res}} - n_0 = \text{distance in field index from resonance}\,.$$

The equation of motion relative to this $n(x, \theta)$ can be solved analytically using the Krylov-Bogoliubov technique (7).

One finds that the resonance buildup of radial oscillations is avoided for

$$\left|\text{radial amplitude}\cdot\left(\frac{\mathrm{d}n}{\mathrm{d}x}\right)^{(1,2)}\right| \ll 8\delta\,. \tag{1.4}$$

($8\delta = 0.2$ in the case of the Italian Synchrotron, and therefore we conclude that this machine lies within very satisfactory limits of stability).

2. – The azimuthally distributed regenerative perturbation.

We admit a beam cross section (at energies above 100 MeV) of about $d = 2$ cm in width by 1 cm in height. The $100 \to 1000$ MeV acceleration time being about $2 \cdot 10^{-2}$ s, the rise time of the proposed regenerative field lattice should be of the same order or shorter.

By making use of the perturbation

$$\begin{cases} \bar{n}(x, \theta) = \left(\dfrac{5}{9} - \delta\right) + \varepsilon \sin 2\theta & \text{when} \quad x > \dfrac{d}{2}\,, \\[2ex] \bar{n}(x, \theta) = \left(\dfrac{5}{9} - \delta\right) - \varepsilon \sin 2\theta & \text{when} \quad x < -\dfrac{d}{2}\,, \end{cases} \tag{2.1}$$

(7) N. M. Krylov and N. N. Bogoliubov: *Introduction to Non-Linear Mechanics* (a free translation by S. Lefschetz from two Russian monographs), Princeton University Press.

$$(2.1') \qquad \bar{n}(x, \theta) = \left(\frac{5}{9} - \delta\right) + \frac{1}{2}\left(\frac{dn}{dx}\right)^{(1,2)} x \sin 2\theta \qquad \text{when} \qquad -\frac{d}{2} < x < \frac{d}{2},$$

($\varepsilon > 0$; $\frac{4}{9} \gg \varepsilon \gg |\delta|$) the desired buildup process can be achieved for radial oscillations without increasing at the same time the axial oscillations. The (2.1) $n(x, \theta)$ required may be get by inserting the $\pi/2$ wide regenerator field shown in Fig. 1 (« $\pm\eta$ » region)

As long as (1.4) is satisfied in the $\pm d/2$ radial extent, the amplitude of radial and axial oscillations remains bounded. Radial deflection begins when $(dn/dx)^{(1,2)}$ is made to increase until (1.4) is violated. The increasing radial oscillation carries the particles into the field lattice (2.1) ($|x| > d/2$). When $|x|_{max}$ differs from $d/2$ by a relatively large value ($|x|_{max}/(d/2) \approx 4$) the equation of motion takes the simple form

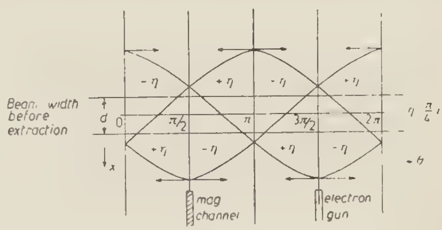

Fig. 1.

$$(2.2) \qquad \frac{d^2x}{d\theta^2} + \left(\frac{4}{9} + \delta\right) x = \varepsilon |x| \sin 2\theta ,$$

and only negligible errors are thus introduced.

Solving this equation after the quoted procedure [7], the first approximation solution is

$$(2.3) \qquad x(\theta) = a(\theta) \sin\left[\tfrac{2}{3}\theta + \Phi(\theta)\right],$$

where

$$(2.4) \qquad \begin{cases} \dfrac{da}{d\theta} = \dfrac{\varepsilon}{2\pi}\, a \sin 3\Phi , & (a(\theta) \geqslant 0), \\[2ex] \dfrac{d\Phi}{d\theta} = \dfrac{\varepsilon}{2\pi} \cos 3\Phi + \dfrac{3}{4}\,\delta . & (0 \leqslant \Phi(\theta) \leqslant 2\pi). \end{cases}$$

The condition to be met to allow an efficient deflection is $|\sigma| < 1$, where $\sigma = (3\pi/2)(\delta/\varepsilon)$. In this case the following results: *Whatever* the initial phase value $\Phi(0)$ is, the oscillation tends to a constant phase value determined by

$$(2.5) \qquad \begin{cases} \cos 3\Phi_\infty = -\sigma , \\ \sin 3\Phi_\infty = +\sqrt{1 - \sigma^2}, \end{cases}$$

that is

$$\begin{cases} \Phi_\infty = 0 & \text{for} \quad \sigma = -1, \\[2ex] \Phi_\infty = \dfrac{\pi}{6} & \text{for} \quad \sigma = 0, \\[2ex] \Phi_\infty = \dfrac{\pi}{3} & \text{for} \quad \sigma = +1 . \end{cases}$$

This means that the maxima of this asymptotic solution can take place only within the azimuthal extent

$$\frac{\pi}{4} \longleftarrow \frac{\pi}{2} \longrightarrow \frac{3\pi}{4} \quad \text{and} \quad \frac{5\pi}{4} \longleftarrow \frac{3\pi}{2} \longrightarrow \frac{7\pi}{4}$$

for $\sigma = +1 \quad \sigma = 0 \quad \sigma = -1 \qquad \sigma = +1 \quad \sigma = 0 \quad \sigma = -1\,.$

Having discussed the stationary regime for the phase, we shall consider now the first equation of (2.4) for the amplitude. One obtains ($\varepsilon > 0$)

$$(2.6)\qquad \begin{cases} a_\infty(\theta) = a(0)\exp\left[\dfrac{\varepsilon}{2\pi}\sqrt{1-\sigma^2}\,\theta\right] & \text{for}\quad \Phi(0) \text{ near to } \Phi_\infty\,, \\[2ex] a(\theta) \;= a(0)\exp\left[-\dfrac{\varepsilon}{2\pi}\sqrt{1-\sigma^2}\,\theta\right] & \text{for}\quad \Phi(0) \text{ opposite to } \Phi_\infty\,, \end{cases}$$

i.e. there is an *unique* stationary behaviour of oscillations, represented by

$$(2.7)\qquad x(\theta) = a(0)\exp\left[\frac{\varepsilon}{2\pi}\sqrt{1-\sigma^2}\,\theta\right]\sin\left[\frac{2}{3}\,\theta + \Phi_\infty\right].$$

It is clear that this is a triple turn extraction mode, and therefore the realizable gain is expressed by

$$(2.8)\qquad g = \exp\left[3\varepsilon\sqrt{1-\sigma^2}\right].$$

By way of example, say $\langle n_0 \rangle = 0.61$, $L/R = 0.335$ i.e. $\delta = +0.025$. In this case in Fig. 2 g and Δ ($=\pi/2$—azimutal point at wich maximum occurs) are plotted versus $\eta = (\pi/4)\varepsilon$. For $n_0 = 0.66$, $L/R = 0.335$ the results relative to g are the same, and Δ must be replaced by $-\Delta$.

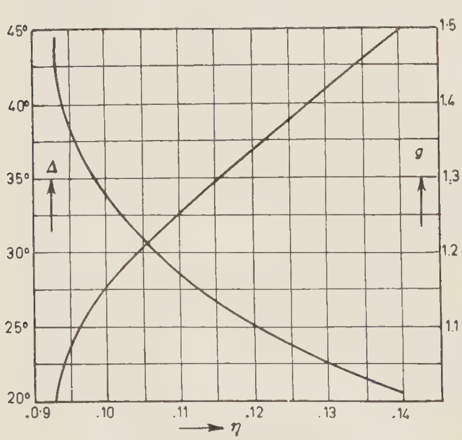

Fig. 2.

From the point of view of axial oscillations one can see that the particle is captured into a motion through a field structure of the form (for the Italian Synchrotron)

$$(2.9)\qquad n \simeq \frac{3}{4} + \frac{\varepsilon}{2}\cos\left[\frac{4}{3}\,\theta - \frac{\pi}{6}\right].$$

The resulting equation of motion may be transformed into a Mathieu equation having divergent solutions ([8])

([8]) N. W. Mc Lachlan: *Theory and Application of Mathieu Functions* (Oxford Clarendon Press).

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only when $\varepsilon > 1.4$. This, however, is not our case and vertical stability of motion is therefore mantained.

The most disturbing terms for this ordinated buildup of radial oscillation (the linear damping term βx and the $1.3 n(x, \theta)$ variation) cannot invalidate the essential aspects of our extraction method when

$$\left\{\begin{array}{l} \left|\left(\dfrac{\mathrm{d}n}{\mathrm{d}x}\right)^{(1,2)} \cdot a(0)\right| \ll 2.2\eta \\ \\ \beta \ll 0.4\eta \,. \end{array}\right.$$

These conditions seem reasonably weak.

Finally, it seems that the proposed perturbation cannot excite other kinds of non linear resonances.

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On High Energy Photoneutrons.

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(ricevuto il 3 Aprile 1958)

Recent experimental investigations on the nuclear photoeffect have shown the existence, at least for some nuclei, of a secondary maximum in the (γ, n) cross section at a photon energy greater than the giant resonance one (1), and of a peak in the energy spectra of the photoneutrons, between 4 and 6 MeV, contributing about 10% to the total yield (2).

Wilkinson's theory (3) does not predict such peaks: in particular the energy spectra of directly photoejected neutrons are expected, according to that theory, to be similar in form to those of the « evaporated » neutrons, with a somewhat longer high energy tail. On the other hand Wilkinson's theory correctly predicts the amount of directly photoejected nucleons (4) but such nucleons may also contribute to the high energy part only of the total yield (5), in disagreement with Wilkinson's prediction on the energy spectra of those particles.

In what follows we will attempt to understand the quoted new features of the nuclear photoeffect in the frame of Wilkinson's theory with the additional hypothesis that directly photoemitted nucleons have to be observed, on the average, with the energy of the virtual Shell Model state whose properties their motion shares just before they leave the nucleus. We will take into account, of course, single particle dipole transitions also when they give a minor contribution to the giant resonance in the photon absorption cross section, but we will choose a somewhat qualitative definition of the relevant single particle level spectrum.

To be more precise about this spectrum, we point out that the basic Wilkinson's assumption, namely that the giant resonance is an effect of the grouping in energy of the $1l$ to $1l+1$ dipole transitions from saturated to unoccupied Shell Model levels, implies that E_m, the giant resonance energy, gives the average energy difference bet-

(1) F. Ferrero, R. Malvano and C. Tribuno: *Nuovo Cimento*, **6**, 385 (1957).

(2) G. Cortini, C. Milone *et al.*: *Experimental results on* Ta *and* Cr *photoneutron spectra* (to be published): private communication.

(3) D. H. Wilkinson: *Proc. Int. Conference on Nucl. Reactions*, Amsterdam (July 1956); *Physica*, **22**, 1039 (1956).

(4) This is implied from the agreement of calculated and experimental ratio of proton emission to total absorption in the heavy elements, as shown in (3) p. 1057.

(5) F. Ferrero, A. O. Hanson, R. Malvano and C. Tribuno: *Nuovo Cimento*, **4**, 418 (1956).

ween the relevant levels, the lower one, in each pair, being known for a given nucleus from Shell Model considerations. The energy of the last filled level is roughly $-E_t$, being E_t the threshold energy either of the (γ, n) reaction, for a neutron transition, or of the (γ, p) reaction for a proton transition.

Furthermore, it is well known that in the Nuclear Shell Model the spin-orbit splitting of l-levels must be taken into account, but this splitting is unessential in Wilkinson's theory: this amounts to say that one can safely assume the unoccupied levels to be splitted in the same way (and with the same strength parameter) as the occupied ones.

The previous remarks strongly suggest to introduce the further assumption that the relevant virtual level spectrum has the same features of the shell model one. This spectrum somehow intermediate between that of the Isotropic Harmonic Oscillator Well and that of the Square Well with infinitely high walls, and with l-levels splitted by spin-orbit coupling, may be assumed as a phenomenological one, in the sense that it gives the ordering and grouping of the stationary single particle levels in terms of which the shell theory explains so many regularities in nuclear phenomena.

With the above outlined scheme, and by considering nuclei with no neutrons outside saturated shells, definite predictions can be given about photoneutrons yields and spectra. Nuclei with A less than 20 will not be considered, being well known that in processes involving such nuclei α-particle effects may be important; nuclei with A around 150 also will not be considered, owing to their usually strong deviation from spherical shape, and to possible effectiveness of collective motions in the dynamics of their structure. In the range of A values between 20 and, say, 110 (^{113}In has a great electric quadrupole moment) some preliminary calculations have been carried out by the present author and it seems that the existence of the peak around 5 MeV in the neutron energy spectra can be reasonably well explained. Such a peak is predicted in fact for ^{40}Ca, ^{51}V, ^{52}Cr, ^{89}Y, usually contributing about 12% to the total yield, and it is expected that nuclei with few protons less or with few neutrons more than those quoted above will not give significantly different results.

Now, in order to test definitely the consistency of such an explanation it is worthwhile to investigate wether departures from the general trend (with the peak around 5 MeV in the energy spectra) outlined just above, are expected on the basis of the present scheme.

This happens to be the case for ^{103}Rh: in fact it may be expected that, working with a bremsstrahlung spectrum with about 30 MeV maximum energy, the neutrons from the $^{103}_{45}$Rh (γ, n) process have a peculiar «fast» peak, in their energy spectrum, at 8.4 MeV contributing something as 6 or 8% to the total yield. The Rh photoneutron spectrum seems to be particularly interesting since the main contribution (84%) to its peculiar «fast» peak comes from the $1h_{9/2}$ direct photoejected neutrons.

The average energy of such neutrons is fixed, according to the present scheme, as follows: the most important contribution to the photon absorption comes from the $1g_{9/2} \to 1h_{11/2}$ transition, and the energy difference between these levels is $E_m = 16.5$ MeV. The spin-orbit splitting of the l-levels is expected to be $\sim(2l+1)\beta$ being (6) $\beta \sim 1$ MeV for nuclei around ^{16}O and $\beta \sim \frac{1}{4}$ MeV for nuclei around ^{208}Pb: so we calculate that the energy difference between the $1g_{9/2}$ and $1g_{7/2}$ levels will be ~ 6 MeV and that between the $1h_{11/2}$ and $1h_{9/2}$ levels ~ 7.25 MeV. The threshold energy

(6) A. A. Ross, H. Mark and R. D. Lawson: *Phys. Rev.*, **102**, 1613 (1956); **104**, 401 (1956).

for the ^{103}Rh(γ, n) process being $E_t = 9.35$ MeV, the energy of the last filled level $1g_{7/2}$ is $E(1g_{7/2}) = -E_t = -9.35$ MeV. For the energy of the $1h_{11/2}$ virtual level one has the value $E(1h_{11/2}) = -9.35 - 6 + 16.5 = 1.15$ MeV what implies that one cannot see any direct emission peak in the photoneutron energy spectra due this level. For the energy of the $1h_{9/2}$ level one has simply $E(1h_{9/2}) = E(1h_{11/2}) + 7.25 = 8.4$ MeV. The amount of directly photoejected neutrons from this virtual level has been calculated according to Wilkinson's theory. According to our choice of the relevant single particle level spectrum, directly emitted neutrons from other virtual levels will have an energy smaller than $E(1h_{11/2}) = 1.15$ MeV or higher than $E(1h_{9/2}) = 8.4$ MeV. The proposed experiment on ^{103}Rh is in progress in this laboratory [7].

* * *

I take a pleasure in thanking hearthly Prof. M. CINI and Prof. G. CORTINI for many interesting discussions on theoretical as well as experimental features of the photonuclear reactions, and Prof. G. CORTINI, Prof. C. MILONE, and their Coworkers for communication and discussion of experimental results [2], in advance of publication.

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(7) Private communication by Prof. G. CORTINI.

LIBRI RICEVUTI E RECENSIONI

G. Boudouris – *Propagation Tropospherique* – Centre de Documentation Universitaire (Place de la Sorbonne, 5) – Paris, 1957.

Il libro in esame riferisce, principalmente, i problemi relativi al calcolo del campo di un'antenna radio (schematizzata, spesso, mediante un dipolo hertziano) tenendo conto dell'influenza della terra, ma trascurando la ionosfera; problemi d'importanza, non solo concettuale, ma anche pratica, perchè con la loro risoluzione è possibile determinare il cosiddetto raggio terrestre e di conseguenza interpretare vari fenomeni delle radio-tramissioni. Il libro appare veramente molto opportuno, perchè su quei problemi esistono poche, e di solito incomplete, vedute d'insieme; inoltre l'A. prende in considerazione numerosi contributi russi, spesso non facilmente reperibili. Il volume è diviso in sette capitoli e in una appendice per le conclusioni. I primi due capitoli hanno carattere introduttivo. Si considerano cioè questioni che potremmo dire classiche, trattate però con dovuta ampiezza e rigore, e cioè: propagazione per onde piane ordinarie ed evanescenti (che l'A. chiama dissociate) in un mezzo omogeneo anche conduttore, fenomeni di riflessione e rifrazione, campo di un dipolo hertziano in un mezzo omogeneo e sue generalizzazioni, il principio di Huyghens, le zone di Fresnel, ecc.

Nei due capitoli successivi comincia la trattazione del campo di un dipolo elettrico e magnetico (schematizzazione di un'antenna a telaio) verticale o orizzontale, immerso in una atmosfera omogenea, supponendo però la terra omogenea e piana, cioè tale da occupare un semispazio. L'A. invece di adoperare le soluzioni esatte di Sommerfeld e Weyl (che espone solo alla fine del cap. IV) per poi procedere, come fanno vari autori, alla loro semplificazione, preferisce, seguendo la scuola russa, partire dalla nozione di condizione al contorno approssimata (su cui ha riferito nel *Suppl. Nuovo Cimento*, **5**, 71 (1957)) ottenendo così direttamente le formule semplificate che si possono usare nei casi concreti.

Il cap. V è dedicato al campo di un dipolo della terra sferica e omogenea, circondata da un'atmosfera pure omogenea. Vengono esposte o riassunte le ricerche più importanti (Poincaré, Watson, Van der Pol e Bremmer) sull'argomento in discorso e non mancano accenni ai lavori più recenti.

Nel capitolo successivo si trattano le questioni più complicate inerenti al caso della terra che si suppone, in generale, ancora piana, ma non omogenea, oppure al caso della terra ondulata per la presenza di colline, fabbricati, ecc. I risultati finora raggiunti, talvolta con metodi semi-empirici, sono esposti opportunamente coordinati.

Infine nell'ultimo capitolo si studia la propagazione qualora l'atmosfera sia resa non omogenea per effetto di differenze di temperatura o di condizioni meteorologiche. È esposta, in modo particolare, la spiegazione dei cosiddetti « duct ». La trasmissione per diffusione troposferica, forse per mancanza di spazio, è appena accennata.

Naturalmente, alcuni degli argomenti sopra ricordati sono esposti solo sommariamente, comunque il libro (opportunamente corredato da grafici, da valori numerici e da un'ampia bibliografia) appare assai utile, non solo per i radiotecnici, ma anche per quei fisici e per quei matematici che desiderano approfondire lo studio dei fenomeni elettromagnetici.

D. Graffi

Year Book of the Physical Society, 1957. The Physical Society, London, pagg. 132, 12 s. 6 d.

L'annuario della Physical Society di Londra, uscito recentemente nella sua edizione 1957, contiene, come d'uso, oltre agli atti della Società, ai bilanci, ai necrologi e ad altre notizie di carattere interno, il testo o il riassunto di numerose lezioni e conferenze, tenute in speciali occasioni.

Il Presidente della Società, N. F. Mott, apre il volume con un lucido discorso sull'impiego di metodi fisici per lo studio dei legami metallici; tra le illustrazioni che accompagnano il testo, sono particolarmente interessanti alcune fotografie di tracce di dislocazione e piani di scorrimento.

Tra gli altri argomenti trattati ricordiamo, l'anno geofisico internazionale (H. Spencer Jones), le meteoriti come fonte di informazione sull'origine del sistema solare (H. C. Hurey), le misure della velocità della luce (L. Essen), i metodi di alta precisione usati presso il National Physical Laboratory per l'esame dei reticoli di diffrazione che hanno permesso di constatare come le proprietà dei reticoli reali siano tutt'altro che uniformi sull'intera superficie (J. Guild), la storia dello sviluppo dei circuiti per il conteggio di impulsi (C. E. Wynn-Williams), la produzione e distribuzione dell'elio liquido in Inghilterra (E. Mendoza) e finalmente la refrigerazione magnetica continua, ottenuta a mezzo di una valvola termica, basata sulle proprietà dei superconduttori (J. G. Daunt).

G. D. Rochester e C. C. Butler riassumono poi ampiamente nelle loro lezioni i risultati dei primi dieci anni di lavoro sugli iperoni e sui mesoni K, mentre le moderne tendenze degli studi nel campo dell'Acustica sono ricordate in una conferenza di E. C. Richardson con particolare riguardo alle ricerche sulla impedenza acustica, sull'origine del rumore aerodinamico dei turbogetti e sul rilassamento molecolare a frequenze ultrasonore.

Franco A. Levi

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